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VARIATIONS IN INSTRUCTIONAL MEDIA, PROCESSES, CONTENT AND APTITUDE VARIABLES IN RELATION TO EFFICIENCY OF COGNITIVE GOAL ATTAINMENT, VOLUME 2.

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APPENDIX I

PROBLEMS OF MEANING AND REFERENCE IN BLOOM'S TAXONOMY: COGNITIVE DOMAIN BY ROBERT WILHOYTE

PROBLEMS OF MEANING AND REFERENCE IN BLOOM'S TAXONOMY: COGNITIVE DOMAIN

BY

ROBERT LEE WILHOYTE

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CHAPTER I

INTRODUCTION: THE PROBLEM IN PERSPECTIVE

Philosophical and Meta Theoretical Aspects

When we see a book called *Taxonomy of Educational Objectives* and subtitled The Classification of Educational Goals[1], our first question is how are these goals defined? Specifically, if goals or anything else are classified, presumably that which is ordered exists, can be referred to, or can be meaningfully defined, etc., before the task of ordering is carried out. If this is the case, are educational goals, on the one hand, discovered and then classified? Or on the other, do we invent goals and then impose some kind of classification scheme on what we have more or less arbitrarily defined?

The general question of what constitutes educational goals has been a recurrent theme in the literature. Most attempts to deal with the problem have centered around the search for, and specification of, content on the criterion of immutability. Thus, early educators defined their goals in terms of the good, the true, and the beautiful. When lack of agreement made these ends questionable, attempts were made to find permanency in human nature or even in social norms. There seemed to be a tendency to think that if immutability were abandoned as a criterion, there would be no reason for saying that education could not be defined capriciously by anyone who wanted to formulate a definition.

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[1] Bloom, B.S., and others, *Taxonomy of Educational Objectives Handbook I: Cognitive Domain*, David McKay & Co., New York, 1956. Handbook I is the first of a series of three. The second one (Affective Domain) has been published. The final volume, as yet unpublished, will be called the *Psychomotor Domain*.

The problems to be discussed in this dissertation are presented in both Handbook I and Handbook II. But all references will be to the *Cognitive Domain* on the assumption that additional documentation would be repetitious.

John Dewey recognized the lack of efficacy in such limited views of the educational process and set forth his well-known definition in which he linked education with growth.

> . . . the ideal of growth results in the conception that education is a constant reorganizing and reconstructing of experience.[2]

Thus, education and growth were seen not as levels or ends to be attained but rather as processes which are self perpetuating.

> Since in reality there is nothing to which growth is relative save more growth, there is nothing to which education is subordinate save more education.[3]

This linking of subject and object provided a basis for understanding the goals of education in terms of process rather than product. The dilemma of deciding between utopian objectives was resolved by substituting the single criterion of growth.

The question which emerges at this point is whether or not "educational objectives" which are to be classified in the taxonomy constitute a body of fixed, inflexible goals.[4] If so, then education loses its dynamic quality. If not then how can goals which are dynamic, emerging, and self reflecting be ordered by and made to correspond to, a classification scheme? This question in a more genral context is the problem of how theory is generated from, and yet explains, that to which it applies. Thus, the relation of theory to application may be taken as defining the general problem area in which this dissertation is located.

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[2] Dewey, John, Democracy and Education, p. 89.

[3] Ibid., p. 60

The specific problem, which must be considered first, is that of explicating just what is meant by the phrase "What is classified" and by the terms "classification scheme" and "taxonomy".[5] An attempt will be made to show that "goals to be classified" and "classification scheme" are the two necessary aspects of the explicative process. The meaning of each is dependent on the other. In order to deal with 'meaning' discursively, the discussion here will have to be about what is explicated. That is, concern is with the meaning of 'meaning'. Therefore, since, the process of abstracting the taxonomy is at the level of theory, a discussion *about* the process may be called meta-theorical. And, just as "classification scheme" and "educational goals" are the two aspects of the explicative process at the level of theory, meta-theory is also divided into the two corresponding areas of meaning and reference. Meaning for the reader is explicated in terms of, or mediated by, the respective contexts within meaning and reference. If the Taxonomy is a scientific work, meaning must be based on what is the case, not on what someone might want to be the case. Yet, we can not just start with raw uninterpreted data. We must have some concept of what an educational objective is in order to be able to identify one. This interaction of meaning (concepts) and reference (relation to empirical data) is what we mean by the term 'explication'. But, throughout this dissertation the terms ' explicating', 'abstracting' and 'generalizing' are used more or less interchangeably.

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[4] One reviewer of the Taxonomy (Affective Domain) raises the question of whether or not a taxonomy without metaphysics is possible. The first section is said to have "metaphysical overtones", to be written in a "dualistic language" and, in general, the authors are said to "...not consistently subscribe to an operational behaviorism". See Morshead, Richard W., "Taxonomy of Education Objectives Handbook II: Affective Domain: Studies in Philosophy and Education, IV: 164-170, Spring 1965.

[5] Some problems to be discussed subsequently are already emerging. When the word 'taxonomy' is used as a name it will be spelled with a capital "T". When it is used descriptively, it will be spelled with a lower case "t". However, to avoid ambiguity, the term 'Handbook I' will be used as often as possible as a shorthand reference for the entire title.

Meta-theoretical terms will be introduced as they are needed and when they are relevant. Connotation, significance, and description will be discussed as sub-areas of meaning, and denotation, naming, and truth will be introduced as sub-topics within reference. The distinction between meaning and reference is treated both formally and functionally. The formal distinction is analogous to the division between logical truth (confirmed by reflection) and empirical truth (verification). The distinction is made for pragmatic reasons. If it is more advantagious, in other contexts, to consider the difference a matter of degree rather than of kind, this can be done. Therefore, in this dissertation, form is subordinate to function. A rather sharp distinction is made at first, for the sake of clarity. But in actual cases of deriving meaning from empirical data, meaning and reference function as two aspects of the same enterprise.

In the remainder of the present chapter specific problems which become apparent after a brief examination of Handbook I are posed in relation to each other. In Chapter II the explication presented in Handbook I is ordered according to the criteria of ostensive, operational and verbal definitions. Chapter III is devoted to a discussion of the three kinds of definitions in terms of connotation-denotation, descriptions-names, and significance-truth. Chapter IV contains a discussion of the explicated taxonomy in relation to criteria stated in the guiding principles. Finally in Chapter V the specific problem is related to the general problem. That is, after an attempt is made to answer theoretical (or meta-theoretical) questions involving the taxonomy as a work of science, the discussion will become more philosophical. The achievement will be examined within the scope of experimentalist philosophy.

Theoretical Aspects

The logical starting point in reading and understanding a work of this kind is to ask what is to be classified and how the classification is to be carried out. The first question is discussed in a section called

What is to be classified?

> Before one can build a classification scheme, it must be clear what it is that is to be classified...It should be noted that we are not attempting to classify the instructional methods used by teachers, the ways in which teachers relate themselves to students, or the different kinds of instructional methods they use. We are not attempting to classify the particular subject matter or content. What we are classifying is the intended behavior of students-the ways in which individuals are to act, think or feel as the result of participating in some unit of instruction. (Only such of these intended behaviors as are related to mental acts or thinking are included in the part of the taxonomy developed in this Handbook).[6]

The specification of what is not to be classified narrows the problem considerably. By eliminating content, the problem of stating educational objectives in terms of what should be taught is avoided. This means that the authors are still not searching for the good, the true, and the beautiful. But if what is to be classified is defined in terms of "intended behaviors", "mental acts", and "thinking", has the problem of defining utopian states of affairs merely been shifted from the context of the known to that of the knower? The next paragraph emphasizes that it is intended behavior which is to be classified and not actual behavior.

> It is recognized that the actual behaviors of students after they have completed the unit of instruction may differ in degree as well as in kind from the intended behavior specified by the objectives. That is the effects of instruction may be such that the students do not learn a given skill to the desired level of perfection; or, for that matter, they may not develop the intended skill to any degree.[7]

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[6] Ibid, pp. 11-12

[7] Ibid, p. 12

Thus, the method of defining educational objectives is "intentional"; the intentions of the teacher, not the actual behavior of students. In other words, educational goals are not defined intentionally by the authors. Instead, teacher intentions are taken as constituting what is given. Questions of how or why teachers decide upon particular education goals are therefore not at issue. What is relevant is merely *that* they agree on certain desired behaviors and what these behaviors are. The subsequent task (that to be taken up in Handbook I) is then that of sorting, ordering and classifying educational objectives as desired cognitive behavior.

To summarize, the educational goals to be classified in the taxonomy are consistent with Dewey's definition of education insofar as each refers to objectives as cognitive operations and not to the content or to the end product of the process. Beyond this point, however, similarity is not clear. That is, the endeavor is empirical with respect to defining educational goals (teacher intentions) and operational with respect to specifying goals with can be meaningfully attained (student behavior). But what criteria do teachers use in stating their goals? Or what justification can be given in support of stated goals? In other words, there is no guarantee that the empirical underpinnings of the taxonomy do not include non-empirical, substantive preferences in the form of teacher intentions which are not supported by empirical and operational criteria.

However, if we accept as given that the educational objectives to be classified in the taxonomy are defined as "the intended behavior of students--the ways in which individuals are to act, think, or feel as the result of participating in some unit of instruction" other questions emerge:

1. How can anything so diverse as educational objectives defined as "teacher intentions" be classified without first making some initial assumptions about mutual exclusion, completeness, internal and external relations, etc.?

2. If principles of ordering are imposed, on the "raw data", do the ordering principles actually define the classes of educational objectives? If this is the case, then do goals stated by teachers function as instances or members of the defined classes?

3. Are there alternative taxonomies? That is, if teacher intentions can in some way be taken as raw data, is the ordered structure (the taxonomy) merely a convenient way of conceptualizing, relating, and explaining the totality of unrelated goals? If the answer is yes, what are the alternatives and on what criteria do we choose between alternatives? If the answer is no, then are the connections between classes logical relations or natural (psychological) relations?

The authors of the taxonomy recognize at least some of these problems:

> Since the determination of classes and their titles is in some way arbitrary, there could be an almost infinite number of ways of dividing and naming the domains of educational outcomes.[8]

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[8] Ibid., p. 13

In order to deal with the problem of arbitrariness, guiding principles are stated.

> First, since the taxonomy is to be used in regard to existing educational units and programs, we are of the opinion that the major distinctions between classes should reflect, in large part, the distinctions teachers make among student behaviors -- a second principle is that the taxonomy should be logically developed and internally consistent.... A third principle is that the taxonomy should be consistent with our present understanding of psychological phenomena... a fourth principle is that the classification should be a purely descriptive scheme in which every type of educational goal can be represented in a relatively neutral fashion... Similarly, to avoid partiality to one view of education as opposed to another, we have attempted to make the taxonomy neutral by avoiding terms which implicitly convey value judgments and by making the taxonomy as inclusive as possible....Another way of saying this is that any objective which describes an intended behavior should be classifiable in this system.[9]

These "guiding principles" may be summarized as follows: the source material is teacher intentions, the taxonomy must not contain contradictions, it must be consistent with psychological research, purely descriptive, and sufficiently broad in scope to include any stated objectives. But these principles are assumptions made by all theoretical research: (1) completeness (there should be no instances which are not explained by the theory). (2) internal consistency (not self-contradictory), (3) external consistency (consistency with related research), and (4) descriptive rather than prescriptive (concerned with what is the case rather than what should be the case).

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[9] Ibid., pp. 13-14

These are necessary criteria or guiding principles for any scientific enterprise, but they are not sufficient for the task of building a taxonomy of educational objectives. What additional assumptions must be made? What rules or requirements must be stated in order to eliminate alternative taxonomies? That is, probably any of the almost infinite number of ways of dividing and naming the domains of educational outcomes would meet the requirements stated in the "guiding principles".

The problem may be summarized in the following manner: on the one hand the raw data are available -- teacher intentions regarding expected student behavior after having completed a unit of instruction. And on the other hand certain necessary guiding principles are available for ordering and classifying the data. But specifically how do the categories emerge? If similarity of behavior on some given criterion constitutes a category, how is the criterion determined? That is, if some other criterion of similarity were used, the data would be ordered in one of the other "almost infinite number of ways".

There are no indications in Handbook I as to how these sufficient logical conditions were met. After stating the necessary conditions (guiding principles), an account is given of how the taxonomy was developed. But the explanations answer the question what was done rather than tell how the gap between the operational and the categorical was closed.

When the task was first started, three categories were selected.

> We initially limited ourselves to those objectives commonly referred to as knowledge, intellectual abilities, and intellectual skills.[10]

Then these three objectives were further sorted, modified and expanded using the operational criterion of simple vs complex.

> We proceeded to divide the cognitive objectives into subdivisions from the simplest behavior to the most complex.[11]

One difficulty in using the simple-complex distinction was that the categories or levels are not distinct.

> A second difficulty in classification results from the fact that the more complex behaviors include the simpler behaviors.[12]

Finally the product of sorting, classifying and relating is presented as a hierarchy containing six levels each of which includes those below it.

> As the taxonomy is now organized, it contains six major classes:
>
> 1.00 Knowledge
> 2.00 Comprehension
> 3.00 Application
> 4.00 Analysis
> 5.00 Synthesis
> 6.00 Evaluation[13]

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[10] *Ibid.*, p. 15

[11] *Ibid.*, p. 15

[12] *Ibid.*, p. 16

[13] *Ibid.*, p. 18

The whole cognitive domain of the taxonomy is arranged in a hierarchy, that is, each classification within it demands the skills and abilities which are lower in the classification order.[14]

Therefore, a student functioning at the level of evaluation, for example, would be domonstrating or using skills which are lower in the hierarchy. But here again problems emerge. Is it possible for a student to perform at a given level without manifesting the skills described at the lower levels? If this is empirically possible, then the classification is internally inconsistent; i. e., behavior at a given level would include skills at the lower levels and at the same time would not include these skills. On the other hand, if it is logically impossible to have a counter instance of that which the taxonomy describes, then the order is not empirically determined: a statement may appear to be descriptive, but actually function as a tautology. For example, we would not seek empirical disconfirmation of the proposition "all fathers are male".

There is evidence throughout the taxonomy that such questions were taken into consideration in the developmental stages, but, again, there is only reference to these problems. There is no descriptive account of how they were resolved. In most cases, the evidence that fundamental logical questions were given due attention is implicit. However, in some cases, the evidence is explicit.

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[14] Ibid., p. 120

We have so far used the terms "classification" and "taxonomy" more or less interchangeably. It is necessary, however, that we examine the relationship between these terms because, strictly speaking, they are not interchangeable. Taxonomies, have certain structural rules which exceed in complexity the rules of a classification system. While a classification scheme may have many arbitrary elements, a taxonomy scheme may not. A taxonomy must be so constructed that the order of the terms must correspond to some "real" order among the phenomena represented by the terms. A classification scheme may be validated by reference to the criteria of communicability, usefulness, and suggestiveness; while a taxonomy must be validated by demonstrating its consistency with the theoretical views in research findings of the field it attempts to order.[15]

Several important questions are implied in this paragraph:

1. If 'taxonomy' and 'classification' are not used interchangeably, what are the similarities and differences? Are the differences definable apart from the context of usage?

2. "Taxonomy . . . has certain structural rules which exceed in complexity the rules of a classification system." What are these rules? How were they applied in selecting the six levels of the taxonomy?

3. "While a classification scheme may have many arbitrary elements, a taxonomy scheme may not. A taxonomy must be so constituted that the order of the terms must correspond to some real order among the phenomena represented by the terms." That is, the taxonomy scheme must in some way be isomorphic with the way the world really is, "while a classification scheme may contain arbitrary elements". Presumably, "classification scheme" means logical classification in which case the requirements would be meaningfulness, consistency, etc., rather than the existence of referents. Thus, in some way, there is a logical structure which is self-contained and superimposed on the raw data of "teacher intentions". In other words, an operational distinction between "classification scheme" and "taxonomy scheme" may be that the referent of the

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[15] *Ibid.*, p. 17

> former is logical entities while the referent of the latter is real entities. This distinction is, however, blurred in the next sentence.

4. "A classification scheme may be validated by reference to the criteria of communicability, usefulness, and suggestiveness; while a taxonomy must be validated by demonstrating its consistency with the theoretical views in research findings of the field it attempts to order". The validation procedure for classification schemes is such as to confirm their logical character (efficacy instead of truth). But if a taxonomy is to reflect the natural order of that to which it refers, then a demonstration of its consistency with the theoretical views of research finding is not adequate validation. Validation would consist in establishing truth as verification which is a stronger criterion than mere consistency.

The above questions constitute a sampling of those which would have to be dealt with in order to properly close the logical gap between the necessary conditions (guiding principles) which must be met in order to develop any taxonomy and the final product (the taxonomy with its six basic levels and subdivisions). This task is that of specifying additional conditions which, when conjoined with the necessary conditions, become sufficient for the generation of the taxonomy which was developed rather than one of the other "almost infinite number" of alternatives.

The fundamental logical problem here is that in any area of theoretical research[16] implicit assumptions are made which are, strictly speaking, extraneous to, and should not become a part of, a particular research work. Nevertheless, these assumptions in all cases are such that, if there are alternatives, the acceptance of an alternative assumption would change the nature of the research product.[17] For this reason, a discussion of the assumptions themselves, along with alternatives, more primitive assumptions, etc., may be at a different level of inquiry (meta-theoretical), and necessitate a discussion which begins by affirming one of the alternative assumptions and from that point on is confined to a particular field of inquiry.

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[16] The term 'theoretical research' is used here to refer to research which is carried out in accordance with some explanatory model or theory as opposed to a more gathering of atomic facts, where assumptions; interpretations, etc., are minimal.

In this case in point the development of Bloom's taxonomy necessitates making theoretical assumptions (here probably meta-theoretical) within the respective areas of *meaning* *and* *reference*. These would have come into focus had the process of going from the guiding principles to the final product been described.[18]

These two areas are referred to elliptically in the discussion of "classification versus taxonomy". "A classification scheme may have many arbitrary elements, . . . be validated by reference to the criteria of communicability", etc. While "a taxonomy must be so constructed that order of terms must correspond to some 'real' order, . . . must be validated by demonstrating its consistency with theoretical views," etc. That is, the criteria for accepting a classification scheme are those of meaning; while, in the case of a taxonomy, the problem is essentially that of reference. Yet, "taxonomies have certain structural rules which exceed in complexity the rules of a classification system". But do these rules merely mirror in some way natural connections which hold in the empirical world, or are they arbitrary means of linking the parts into a system in order to present, understand, and explain that which is ordered? If the latter is the case, then there are problems of meaning in taxonomies as well.

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[17] Familiar examples are the assumptions of mathematical statistics (normality, continuity, infinity, etc.) which must be made when applying statistical designs in research.

[18] The omission of this account can be justified on the basis that the purpose of the taxonomy is to provide educators with the product, and the genesis is therefore irrelevant; i.e., readers are concerned with *how to classify educational objectives and not with how to obtain a method for classifying educational objectives*.

The only conclusion seems to be that differences between taxonomies and classification schemes are differences of degree rather than of kind. Criteria for generating taxonomies are mainly those of reference rather than meaning, while in the case of classification schemes criteria of meaning are more applicable than those of reference. The levels and sub-levels in the taxonomy are not merely names (reference); nor are they merely descriptions (meaning). Definitions are not only intentional (meaning); nor are they simply extensional (reference). The validation procedure is not only seeking the truth as coherence (meaning); it is also the establishing of truth as correspondence (reference) etc.

The specific problem to which this dissertation is directed is therefore that of closing the logical gap between the guiding principles and the completed taxonomy. Of course, the entire Handbook is indirectly an effort to do this. But since the main intention is to provide educators with a useful instrument, the problem of development is marginal rather than focal. Also, continuity of development is difficult to establish when the criteria (structural rules, validation procedures, etc.), are extraneous to the purpose for which the work is written.

The analysis will thus begin with a discussion of the function of definitions in the taxonomy, (basically there are three types). The next step will be to bring into focus assumptions which are within the area of meaning and to distinguish them from correlative assumptions in the area of reference and conversely.[19]

More specifically, problems of meaning and reference will be introduced through a discussion of the function which definitions perform in generating the taxonomy. Contrary to the manner in which definitions are sometimes used, they do not constitute the formation rules for structuring the content of a given field. That is, when a definition is stated, certain assumptions are necessarily made about the subject matter in question within a context of meaning-reference. Since these assumptions are, at best, only implicit, any alternatives and consequences of alternatives are not available for consideration. Another way to state this point is to say that since the function of each type of definition is invariant with respect to different theoretical areas of content, definitions themselves are therefore metatheoretical and serve as the connecting link between theory and metatheory.

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[19] For example, consider the statement "Evaluation is defined as the making of judgments about the value, for some purpose, of ideas, works, solutions, methods, materials, etc."

If the phrase "is defined as" is replaced by either "means" or "refers to", the meaning of either statement may appear to be the same. However, when "means" is used in the process of defining the reader should expect an intentional definition; whereas, the use of "refers to" indicates that the writer is defining a term ostensively; i. e., presenting the reader with a sample from the class of situation or objects to which the term applies.

After distinguishing each type of definition in terms of how it functions in the context of the taxonomy, attention will shift to the level of meta-theory. The function of ostensive definitions will be shown to be that of presenting meaning as instances to which a term applies. Operational definitions are present as implicit rules which restrict what is relevant (i.e., only one response in the empirical context of a multiple choice item is considered correct). Finally, verbal definitions present meaning by equating the term to be defined (definiendum) with a verbal statement of what the term means (definiens). The purpose of verbal definitions is to convey meaning at an abstract, symbolic level. If empirical reference is not clear, a verbal definition may be meaningless. Thus, it must be shown how, or under what conditions, meaning (empirical significance) is presented in instances ostensively defined, how meaning is restricted by rules which operate within the presented context and how meaning is then reduced and stated as the defining characteristics (connotations) of a verbal definition.

There are at least two other alternative methods of presentation which could have been employed. One would be to present a survey of the fields of meaning and reference and to show how the content of these fields is applied in generating the taxonomy. Another method would be to deal separately with each instance or situation in the taxonomy structure involving issues of meaning and reference; i. e., to have a division according to topics, and to explicate each instance separately.

The first alternative was rejected because it is unnecessary to present a complete survey of the respective fields of meaning and reference in order to explicate problems and assumptions in the taxonomy. The second alternative was rejected because it is essentially a method of disecting the product and would not focus on the process of going from the guiding principles to the completed taxonomy. Also, it would be repetitious insofar as the problems of defining are much the same regardless of the content involved. In other words, the different instances are best employed for explicating and illustrating the problems rather than defining them.

Just as no subject is sui generis but is dependent upon assumptions which take the form of definitions, the subject of definitions as meta-theory (meaning and reference) is itself not self-contained. A residue of problems in the development of the taxonomy will remain and emerge when attempts are made to meet the completeness criterion (one of the guiding principles). For example, prima facie the statement "X is a knowledge level statement" is itself at the knowledge level. But, if so, it will be shown that a contradiction is generated; i. e., the consistency criterion is violated. Hence, certain additional assumptions must be made.

Problems of meaning and reference taken separately and in the simplest sense are merely problems of validation. We accept an abstract system of mathematics on meaning criteria alone; and we accept verbal explanations of phenomena because they correspond to the objects and events to which they refer. Most (probably all) scientific enterprises include these two fundamental dimensions. Bloom's Taxonomy was chosen for emphasis in this dissertation because the problems are more focal, more concentrated, and better represented, and also because it is outstanding in the field.

CHAPTER II

THE FUNCTIONS OF DEFINITIONS IN THE TAXONOMY[1]

The Need for Definitions

The central place of definitions in inquiry was recognized by William James when he wrote the following words:

> Some years ago, being with a camping party in the mountains, I returned from a solitary ramble to find everyone engaged in a ferocious metaphysical dispute. The corpus of the dispute was a squirrel - a live squirrel supposed to be clinging to one side of a tree-trunk; while over against the tree's opposite side a human being was imagined to stand. This human witness tried to get sight of the squirrel by moving rapidly round the tree, but no matter how fast he goes, the squirrel moves as fast in the opposite direction, and always keeps the tree between himself and the man, so that never a glimpse of him is caught. The . . . problem now is this: Does the man go around the squirrel or not? He goes round the tree, sure enough,

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[1] An underlying premise in the dissertation in general and in this chapter in particular is that form is subordinate to function. The reason for using three types of definition as criteria is not that inquiry necessarily corresponds to any particular form, but rather because each serves a special function in the explication of meaning.

Hullfish and Smith, in the context of analytic-synthetic, note how confusion can arise "when we fail to recognize the authority of function over form . . ."

> A main difficulty with all of the usual conceptions of the analytic-synthetic classification is that these terms are taken to designate types of statements rather than types of functions which statements perform in inquiry. When we fail to recognize the authority of function over form, then the force of 'must' (in the statement "The analytic is about what must be true . . .") is attributed to logical necessity. But where does logic get its necessity? Surely statements which are true by logical necessity are so because they are analytic. We thus arrive at a kind of vanishing point . . .

See Hullfish, H. Gordon, and Smith, Philip G., Reflective Thinking: The Method of Education, p. 80

> and the squirrel is on the tree; but does he go round the squirrel?[2]

James points out that before we can answer the question, we must first specify what we mean by certain key terms:

> "Which party is right," I said, "depends upon what you practically mean by "going round" the squirrel. If you mean passing from the north of him to the east, then to the south, then to the west, and then to the north of him again, obviously the man does go round him, for he occupies these successive positions. But if on the contrary you mean being first in front of him, then on the right of him, then behind him, then on his left and finally in front again, it is quite as obvious that the man fails to go round him, ... make the distinction, and there is no occasion for any farther dispute. You are both right and both wrong according as you conceive the verb 'to go round' in one practical fashion or the other.[3]

This famous problem posed by William James illustrates the function which definitions serve in the structuring of problematic situations. Such questions as "Is light really a wave motion or a particle function?" "Do I.Q. tests really measure innate capacity for learning?" "Does the squirrel really go round the tree?" imply that subsequent inquiry would yield a yes-no, true-false answer. The possibility that confusion is built into the questions frequently has been overlooked. The structure of the universe is thought to be intelligible and knowable. Science consists in discovering universal truths. James was pointing out that seeking yes-no, true-false answers to cetain types of questions can be at least non-efficacious and at most confusing. But there is also a corresponding danger to avoid when we follow the mandate to define our terms. Does the freedom implied in the act of defining terms mean that 'meaning' and 'truth' are whatever we decide they should be? James pointed out that there are no restrictions placed on how we define our terms, that the meaning of such phrases as "to go round" is not the same in different places and situations. Yet some meaning is fixed; i. e., we do not have the choice of assigning any meaning we wish when we specify the meaning of key terms in a given situation.

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[2] James, William, Pragmatism, p. 41.

[3] *Ibid.*, p. 42

Logically the first step in developing the taxonomy is that of specifying what is to be classified. Hence, at this early stage, the authors were confronted with a problem of definitions. What is to be classified? Educational goals. How are educational goals defined? As teacher intentions. How are teacher intentions defined? As expected student behavior. How is expected student behavior defined? As "the ways in which individuals are to act, think, or feel as the result of participating in some unit of instruction". But what is meant by "the ways individuals are to act, think, or feel ...". The function of definitions is to convey meaning. Prima facie, the above quotation appears to raise more questions than it answers. The next two sections in the text compound the confusion.

> Only such of these intended behaviors as are related to mental acts or thinking are included in the part of the taxonomy developed in this Handbook.[4]

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[4] Bloom, op. cit., p. 12.

And in the next paragraph:

> It is recognized that the actual behaviors of the students after they have completed the unit of instruction may differ in degree as well as in kind from the intended behaviors specified by the objectives.[5]

"What is to be classified" is therefore intended student behaviors which are related to mental acts and intended student behavior which may never be attained either in degree or in kind in actual student performance.

An impasse seems to have been reached. It is apparently then necessary to define "mental acts", how student behavior is related to mental acts, and how intended behavior is related to actual behavior. In other words, if educational objectives, defined as intended student behaviors, are in some way related to mental acts, and are goals for which students may strive but not attain, it seems that the function which definitions are supposed to accomplish (convey meaning) has not been met. On the one hand, there seems to be a circularity of meaning ('educational goals' means 'teacher intentions' means 'intended student behaviors", etc.) and on the other, there appears to be the danger that educational objectives may be defined in terms of non-verifiable propositions which are non-efficacious and misleading. The goal of education is more education rather than the achievement of desired mental states or acts intended by the teacher.

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[5] *Ibid.*, p. 12

Before this impasse can be resolved, it is necessary to briefly examine types of definitions in a simple context. We might convey the meaning of 'table' non-verbally by merely pointing to a table. Or we could say "this is a table". On the other hand, we could define 'table' as "a flat surface mounted on three or more legs". Still another alternative would be to say that a table is any surface used for the purpose of serving food. None of these definitions completely conveys the meaning of 'table'. In the first example all of the accidental properties in the one instance referred to would be incorporated into the meaning of 'table'. In the second case, the question arises if any surface mounted on three or more legs is a table. And, in the third case, is the function for which an object is to be used either a necessary or a sufficient condition for conveying the meaning of the term which is the name for the object? Each type of definition will leave unanswered questions. Yet, depending on the nature of what is to be defined, one type may be preferable to another. In some cases each type may be employed.

In Handbook I descriptive statements, factual assertions, names, etc. may all function as definitions. Therefore, before proceding further in explicating the meaning of "what is to be classified", a short discussion of each type of definition is necessary.[6]

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[6] The distinction here is primarily functional. In chapter III, a more formal, meta-theoretical comparison is made.

Ostensive Definition

Probably the simplest way to define a term is to refer to an object denoted by the term. Hospers emphasizes the non-verbal character of definitions.

> Ostensive definition, as the name implies, shows you, confronts you with an instance or instances of the word's denotation. Showing someone a birch tree would be giving him an ostensive definition of the term "birch tree". No words except "birch tree" (with an act of pointing) need to be used.[7]

The persistent use of verbal definitions to convey meaning involves a circular process ('educational objective' means teacher intentions, 'teacher intentions' means expected student behavior, etc.). In order to avoid circularity, it is necessary at some point to go to ostensive definitions.

> It would seem that the meanings of at least some of the words in a language have to be indicated ostensively. Suppose for a moment we have only verbal definitions. Each word is defined, and in doing so, other words are used. This will help us only if we already know what these words mean. How will we find out what they mean? By having their meanings explained by the use of still other words. And so on. But how can this process go on forever? Must we not eventually come to the point where we directly connect words with things, and not with other words, lest we be caught up forever in the circle of our own words? -- To connect words with the world, we need ostensive definition.[8]

The use of verbal definitions presupposes a knowledge of the meaning of other words. When children begin the slow process of learning what words mean, they learn that a particular word is a name used when referring to a particular object (one instance of the word's denotation). The learner must then look for other instances which are like the first one and finally decide what characteristics of an object or situation are essential if a term is to be applied.

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[7] Hospers, John, An Introduction to Philosophical Analysis, p. 56

[8] Ibid., p. 56

> . . . someone asks, "What is an epic poem?" and is answered, "The Iliad, The Odyssey, the AEneid, Paradise Lost, and anything like these". The difficulty is to know "like in what respects?" . . . The answer does not take us far but it is a beginning.[9]

In employing ostensive definitions to convey meaning it is not necessary to retrace the entire process of concept formation (to commit the genectic fallacy). Rather, the meaning of the term must be explicated. Readers of the Taxonomy will attach some meaning to the term 'educational objectives'. That is, they are not learning the meaning of a nonsense syllable. But the meaning must be clear and nonambiguous if the "guiding principles" are to be met. And it will be shown that ostensive definition is "a beginning".

Verbal Definitions

The term 'verbal definitions' is used here to refer to those definitions which convey meaning by substituting another term or terms for the word in question. ('Educational goals' means teacher intentions, 'table' is a surface mounted on three or more legs.)[10] Stebbing states as rules the requirements which verbal definitions must meet.

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[9] Stebbing, L. Susan, A Modern Elementary Logic, p. 118

> Let us first consider the traditional rules, which presupposes that what is needed is an explanation in words of how a given word is to be understood. The word to be defined is tradionally called the _definiendum_, the defining phrase is called the _definiens_.
>
> A. Rules concerned with the nature of definition.
>   1. The _definiens_ must be equivalent to the _definiendum_. From this rule two corollories follow:
>   (1.1) The _definiens_ must not be wider than the _definiendum_.
>   (1.2) The _definiens_ must not be narrower than the _definiendum_.
>
> B. Rules concerned with the purpose of definition.
>   2. The _definiens_ should not include any expression that occurs in the _definiendum_ or that could be defined only in terms of it.
>   3. The _definiens_ should not be expressed in obscure or figurative language.
>   4. The _definiens_ should not be negative in significance unless the _definiendum_ is primarily negative in significance.[11]

The purpose of giving a definition is " . . . to make clear the limits within which a word, or phrase, can be rightly used."[12] It must be understood that attempts to meet the requirements set forth in the above rules are attempts to give reportive or lexical definitions. We are reporting on the way a term is used in general. We are not stipulating how we are using the term. On the other hand, we could be stating what we are going to mean by it, with the implicit understanding that although the reportive and stipulative definitions are not necessarily the same, each is consistent with the other.

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[10] In the next section it will be shown that operational definitions are verbal insofar as words are used to communicate that to which the term refers. But the referents are operations or processes which, like ostensive definitions, are not intended to provide the complete meaning of the term in question.

[11] Stebbing, L. Susan, _op. cit._, p. 119

[12] _Ibid._, p. 119

In most cases it is difficult to meet the requirements set forth in these rules. The definiendum 'table' and the definiens ' a flat surface mounted on four or more legs' are probably not interchangeable. That is, we might wish to call some flat surfaces mounted on three or more legs 'stools' or'desks'. However, the definiendum 'triangle' is equivalent to the definiens 'three sided plane figure'. So at least in some cases the criterion of interchangeability can be met. Also, the fact that we cannot satisfy the conditions stated in the rules merely emphasizes the function which rules or standards serve; to specify conditions which we attempt to meet but which may only be satisified to some degree in any given case.

Attempts to give verbal definitions are usually made in accordance with these rules and are of the form per genus et differentiam; i.e., by assigning the genus and distinguishing characteristics. The definition of 'table' is incomplete because the distinguishing properties are not complete. The definition is broad enough to include stools, desks, etc.

In a similar manner, to define 'educational objectives' as 'teacher intentions' would be too broad. A teacher may intend many states of affairs which would be difficult to defend as educational goals. The distinguishing characteristic 'student behavior' limits meaning somewhat, but the added predicate "ways in which individuals are to act, think, or feel as the result of participating in some unit of instruction" limits meaning still further. The question then becomes, Does this definition satisfy the criteria for good verbal definitions? That is, can the definiendum 'education goals' be replaced by the definiens "intended student behavior - ways in which individuals are to act, think, or feel, etc"? If meaning is completely conveyed, then the reader should understand just what is to be classified. On the other hand, the definition may be of the form "bachelor means unmarried male" in which case if the reader does not understand the meaning of the definiendum he most likely also does not understand the meaning of the definiens. Probably, the definition is intended to completely convey meaning; otherwise the guiding principle of completeness would not be met. In other words, if there is a surplus of meaning to 'educational goals' which is not conveyed, then this meaning would be excluded from consideration in the taxonomy and, a fortiori, "that which is classified" would not include all possible teacher intentions.

Verbal definitions can be used in the total process of explicating the meaning of "what is to be classified", but there are certain unavoidable limitations. An attempt will be made to show how this type of definition is employed in the taxonomy and also to point out how the task of the taxonomy (defining what is classified and how the classification is carried out) is necessarily limited by the extent to which verbal definitions do not perform the entire function of conveying meaning.

OPERATIONAL DEFINITIONS

The meaning of such terms as 'think', 'know', 'believe', etc. are difficult to define. How can we establish that X is thinking? Wittgenstein's famous dictum that the meaning of a term is in its method of verification was an attempt to deal with such matters. A similar method of establishing and conveying meaning is the specification of operational definitions. Bridgman's well known formula, "In general, we mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations"[13] may be an overstatement if he is referring to all concepts. However, as Hempel points out, the meaning of such terms as "harder than" can be specified operationally with little difficulty.

> For example, the term 'harder than' might be operationally defined by the rule that a piece of mineral, X, is to be called harder than another piece of mineral, Y, if the operation of drawing a sharp point of X across the surface of Y results in a scratch mark on the latter.[14]

Bechtoldt stresses both the form and the function of operational definitions.

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[13] Bridgman, P. W., The Logic of Modern Physics, p. 6.

[14] Hempel, C. G., "A Logical Appraisal of Operationalism", Scientific Monthly, 79: 215, October, 1954.

> An operational definition is simply a verbal statement of the if-then type specifying the observable conditions or rules of procedures under which the term is to be applied in the definition of descriptive or empirical concepts or variables.[15]

Hempel's example, translated to an if-then statement, would become, If mineral X is harder than mineral Y, then the former will scratch the surface of the latter. The function of operational definitions (specifying operational conditions or rules . . .) may be better understood if a few specific cases are examined. For example if we wanted to define 'table' we probably would not find it necessary to specify the operations in which one engages in order to construct a table. An ostensive definition (pointing to an instance) or a verbal definition (substituting a description for a name) would be easier and more effective. But how would we go about pointing to an instance of "intelligence" in the clear, unambiguous way in which we can point to a table? Or how would we go about describing "intelligence"? We may succeed in giving ostensive and verbal definitions, but the task of conveying meaning might better be accomplished by specifying the kinds of operations or processes to which the term "intelligence" would be applied.

In general, a distinction can be made between words like 'table' and words like 'intelligence' by noting that 'table' refers to a class of objects which are directly observable, while the referent of 'intelligence' is not an object which is observable in the same way. We may point to an instance of intelligent behavior, but we cannot point to an instance of intelligence itself. In general, then, the function of operational definitions in inquiry is to convey meaning when it is either difficult or impossible to refer to the extension of the term either by pointing (ostensive definition) or by description (verbal definition). Bergmann, in discussing what he calls "Bridgman's formula", points out that reference is not central.

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[15] Bechtoldt, H. P., "Construct Validity: A Critique", Psychological Bulletin, 14: 621, October, 1959.

> His (Bridgman's) formula lends itself to still another interpretation. Taken literally, it (operational definitions) identifies the meaning of a concept not with its referent, but with the operations one must perform if one want to test the presence of this referent.[16]

Terms which are defined operationally may have referents, but the emphasis in explicating meaning is on the operations which must be performed in order to test the presence of referents rather than on the referents themselves. Bechtoldt states that constructs used in psychology have both referents and rules for their use.

> Terms like hostility, aggression, psychopathic deviate, intelligence and mechanical ability have some referents and some rules for their use.[17]

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[16] Bergmann, Gustav, "Sense and Nonsense in Operationalism", Scientific Monthly, 79: 214 (1954). It is interesting to note that operational definitions were first employed in order to be able to say that such hypothetical constructs as 'election' were meaningful although their existence could not be guaranteed; that is, the referent of 'election' could not be directly observed.

[17] Bechtoldt, op. cit., p. 621

The question of what is meant by 'educational objectives' may be rephrased as to what does 'educational objectives' refer? By doing this, we avoid the apparent circle from the name to the description "ways in which individuals are to act, think, or feel as the result of participating in some unit of instruction." But the referent is not clear. That is, just what would count as an instance of the specific intended behavior of students? Or, to put the question another way, how do we verify that particular educational goals have been met? How do we know that a student has knowledge? that he can function at the levels of comprehension, application, etc.? Because he can respond correctly to a test item which represents that particular cognitive level.

Since approximately half of Handbook I is devoted to a classification of test-items, educational goals, as levels within the cognitive domain, are implicitly defined in terms of operations which, if performed correctly, permit the inference that certain educational goals have been met. What is knowledge? The operation of responding correctly to test items which represent this cognitive level. This operational definition may appear circular, but consider the alternative of looking for the referent itself. How would we verify the presence of knowledge? Not by brain surgery; not, by measuring brain waves; not by electrochemical analyses. There is no other acceptable alternative to using student behavior as the criterion, and this entails stating operational definitions.

The remainder of the present chapter will be devoted to examining

what is to be classified and how the classification is to be done using the three types of definition discussed as criteria in accordance with which meaning is conveyed. An attempt will be made to show that all instances of conveying meaning in the explication presented in Handbook I are within the range of these three kinds of definition. After the modes of presenting meaning are crystalized out, each type will be discussed separately so that the necessary assumptions and limitations which are entailed can be pointed out. It will be shown that these restrictions are manifested in the context of the taxonomy where definitions are employed.

It is not expected that the gap between the guiding principles and the completed taxonomy will be tightly closed; that is, that there will be no unanswered questions. But the structure provided will have two advantages:

(1) The assumptions made will be explicit rather than implicit.

(2) The assumptions will be at the level of meta-theory rather than at the level of theory.

Organization of the Taxonomy for the Explication of Meaning

The development of the taxonomy moves abruptly from an outline of the problem in Chapter 1 to an application of the completed classification scheme in Chapter 3.[18] The following items were accomplished

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[18] **Chapter 2 is devoted to a tangential discussion of curriculum, and is therefore extraneous to the central task of presenting the taxonomy as a meaningful and useful scheme which meets the criteria stated in the guiding principles. (See Appendix A.)**

in Chapter 1.

1. 'Taxonomy' was defined
2. 'What is to be classified' was defined.
3. Guiding principles were stated.
4. A distinction was made between 'classification' and 'taxonomy'.

In Chapter 3, the reader is provided with an opportunity to classify educational objectives and test items. Meaning is thus explicated in the act of doing.

> The intent of this chapter is thus to bring the reader closer to the taxonomy and to convert it from an empty set of terms and definitions to one that the reader finds workable and applicable to his own situation.[19]

In other words, the terms used in defining and describing the classes and sub-classes may be logically significant but they may not necessarily convey meaning to the reader. Thus, the explication in Chapter 3 presupposes the entire taxonomy which is defined, described and illustrated in part II. Otherwise, the examples would only be defined ostensively. That is, the reader would have to abstract the common properties, or meaning, which is provided in the "empty set of terms and definitions". Chapter 3 therefore functions for the reader as a test situation: as an opportunity to ascertain if what has been read in part II can be applied. Logically speaking, then, Chapter 3 is the final chapter.[20]

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[19] Bloom, op. cit., p. 45

The Taxonomy as Instances of Three Types of Definition

The first part of Chapter 3 contains a meta-theoretical discussion of the ways in which meaning is conveyed in part II. Three levels of definition are used in each taxonomy category.

> The taxonomy Handbook in its present form defines a class or sub-class of educational objectives in three ways. The first and major type of definition is represented by a verbal description or definition of each class and sub-class . . . A second type of definition is provided by a list of educational objectives which are included under each sub-class of the taxonomy . . . The third type of definition attempts to make clear the behavior appropriate to each category by illustrations of examination questions and problems which are regarded as appropriate.[21]

The first type of definition is therefore verbal: the statement of equality between the definiendum and the definiens. If the words which constitute the definiens are no more meaningful than the definiendum, then the term is "empty" - no meaning is conveyed. For example, "Synthesis is here defined as the putting together of elements and parts so as to form a whole".[22] If the terms used in the definiens are significant for the reader, then meaning is conveyed.

The second type of definition is accomplished by stating illustrative educational objectives taken from various sources in the literature. "Ability to formulate a theory of learning applicable to classroom teaching" and "ability to make mathematical discoveries and generalizations"[23] are examples of derivation of a set of abstract relations, a sub-class of synthesis. This type of definition is thus either ostensive or operational depending on the function it serves in conveying meaning. If a description is given of one instance of the denotation of the term, and the reader has to abstract the meaning by determining what the separate instances have in common, then the definition is ostensive. On the other hand, if meaning is conveyed by stating the condition for the truth of a sentence in which the term occurs, then the definition is operational. For example 'Extrapolation' is the ability, or process, or operation of "predicting continuation of trends."[24]

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[20] This point is important and, in the broadest sense, the remainder of the chapter will be devoted to its support.

[21] Bloom, op. cit., p. 44.

[22] Ibid., p. 162

The third type of definition consists in test items which represent or illustrate a particular class or sub-class. The following item is an example of translation, a sub-class of comprehension.

> A group of examiners is engaged in the production of a taxonomy of educational objectives. In ordinary English, what are these persons doing?
>
> A - Evaluating the process of education
> B - Classifying teaching goals
> C - Preparing a curriculum
> D - Constructing learning exercises

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[23] Ibid., p. 172

[24] Ibid., p. 96.

This exercise involves - translation of "taxonomy" to "classification", a less abstract term. There is no context clue.[25]

As in the second type, this method of definition may function either operationally or ostensively. 'Synthesis' is the process of responding correctly to a test item which represents this class of educational objectives. Or, on the other hand, if the test item is presented as an instance of synthesis, the "definition" functions ostensively.

The remainder of Chapter 3 consists in the presentation of sample educational objectives and sample test items which the reader can classify in order to test the degree to which the meaning of the taxonomy has been explicated for him. If the meaning of the illustrative examples is abstracted, and if the verbal definitions presented are significant, then the task of conveying the meaning of what is to be classified and how the classification is to be carried out is complete.[26]

The meaning of "What is to be Classified"

The above brief description of the manner in which meaning is explicated in the taxonomy and the functions which definitions serve in the process still leaves unanswered just what it is that is to be classified. However, now a much clearer statement of the problem can be made.

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[25] Ibid., p. 99

[26] Questions of consistency, completeness, etc. (conditions stated in the guiding principles) would, of course, still be unanswered. The reader may have classified ten or ten thousand test items, but how does he know that the next item will be classifiable? i. e., what is the test for completeness?

The attempt to state a verbal definition of "what is to be classified" resulted in an impasse.[27] Reasons will now be given for the lack of clarity in defining "educational objectives", an explication will be presented of the ways in which meaning is conveyed, and questions will be raised as to the adequacy of these modes of presentation.[28]

Terms which refer to physical objects can be easily defined ostensively. Also verbal definitions can be given when the properties refer to observable characteristics. However, terms like 'mass', or 'intelligence' do not refer to any objects which can be directly observed. Brodbeck contrasts terms which have observable referents and those that do not, and notes the central importance of operational definitions for the letter.

> The characteristic abstractness of scientific concepts, like mass or IQ, lies in the fact that these terms cannot be defined by simply listing a cluster of directly observable attributes. Merely by looking at a surface we can tell whether it is red or by looking at an object whether it is a gnu. We cannot so simply tell what the mass of an object or the IQ of a child is.

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[27] "Educational objectives" means teacher intentions, "teacher intentions" means intended student behavior, and the latter means "the way students are to act, think, or feel as the result of participating in some unit of instruction".

[28] For a clear and concise discussion of the ways in which modes of meaning function in inquiry, see: Grandstaff, Marvin, Logical Aspects of Critical Mindedness, Unpublished doctor's dissertation, I. U., 1964, pp. 33-48

> Yet a body has mass as well as color; a child has an IQ as well as blond hair. However, more complicated observations are required to know that a body has a certain mass than to know that it has a certain color. This greater complication is reflected by the way the concepts of science are defined, for scientific definitions are rarely of the simple, dictionary type . . . They are not, precisely because they name features of the work that can be discerned only under certain conditions and these conditions are part of the meaning of the concept. They must therefore be included in the defining properties of it. To do this requires an operational definition.[29]

What is the referent of "educational goal"? Since Handbook I is a classification of educational goals in the cognitive domain,the objectives referred to involve the thinking process. But how does one define levels, or degrees, or kinds of thinking? This can only be done thorugh verification. That is, if there is no way to verify the claim, "X is thinking", then it is meaningless to assert that X is now engaged in a cognitive process called thinking. "Thinking" then becomes coextensive with "thinking acts" and the latter term refers to the behavior of X under certain specified and known conditions. "What is to be classified" is intended student behavior, but the behavior which the teacher intends is not cognitive processes or mental states which can not be verified. The referent of such terms as 'comprehension' is the expected behavior of students on certain kinds of test items. And expected behavior is the expectation (intended, not actual) that students will respond correctly.

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[29] Brodbeck, May, "Logic and Scientific Method in Research on Teaching", in Handbook of Research on Teaching, Edited by N. L. Gage, p. 49

Terms used to name classes and sub-classes in the taxonomy are therefore constructs like 'mass', 'I. Q.', 'psychopath', etc. They refer to operations which can be carried out rather than to objects which can be directly observed. They are, therefore, defined operationally. On the other hand, if a term such as 'synthesis' refers to certain kinds of situations, then the term denotes the situations and the situations constitute the denotation of the term. To point, or to in some other way call attention to one instance to which the term applies is thus to define the term ostensively. Particular goals, constructs, may be defined either ostensively or operationally in the taxonomy depending on the function which a given sentence serves in the total process of conveying meaning.

The referents of terms used in type III definitions are test items. But what are the referents of terms used in instances of type II definitions? The reference of "educational objectives" is broader than that of "test items" and includes the latter. Type II definitions therefore refer to properties or characteristics which test items have in common, i. e., to classes of test items. But when definitions of the second type are formulated, the common property or characteristic of test items is expressed in terms of an ability to respond correctly to items which constitute the class. The ability to read musical scores and the ability to read architectural plans are two types of instances of translation - a sub-class of comprehension (p. 92). These and other abilities denoted by 'translation' are finite. Otherwise the criterion of completeness is, by definition, not met. The possible instances of translation are infinite but the kinds of translation must be finite. Another way to make the same point is to say that the number of situations to which the term 'translation' may refer is infinite, but the number of ways these situations can be characterized or described is finite. Or, in terms of definitions, it must be possible to give a finite definiens, otherwise the rule for definitions, that the definiens be equivalent to the definiendum could not be met. This equivalence is also necessary for type I definitions where one term, the definiendum, is a general term which is sufficiently broad to encompass in its meaning all of the descriptions presented in type II definitions.

Returning to the question of reference in regard to type II definitions, do descriptions have referents? This question will be discussed in detail in the next chapter. At this point, it is only necessary to note that a statement of the form, "The present king of France is bald" may have no referent but is, nevertheless, meaningful. The descriptions given in type II definitions refer to intended student behavior. If this "behavior" refers indirectly to correct responses on test items, then test items are the referents. But since educational objectives stated as descriptions are broader and more general than the mere presenting of test items, emphasis is on meaning rather than reference. Thus, for the reader who is attempting to understand the descriptions which constitute type II definitions, the focus is also on meaning rather than reference. Presumably, when someone describes 'translation' as the ability to translate as read musical scores, the assumption is made that there are test items which represent this sub-class of translation and that there are students who can correctly respond to these items. But the descriptions would be meaningful even if there were no students who possessed the ability to do this kind of translation. In other words, it may be the case that no student responds correctly to the educational goals described in type II definitions and represented by test items in type III definitions. Educational goals represent teacher intended student behaviors, not necessarily actual behaviors. Some or no students may respond correctly, but the teacher can still intend or expect correct responses. Probably no teacher would describe intended student behavior if he did not expect some students to behave in the intended manner. However, what is important here is that the existence of a referent is not a necessary condition for meaning in the case of descriptions. Regardless of whether educational objectives refer to test items or responses to test items, the problem of understanding a description is a matter of meaning as well as reference. But meaning is possible even if there is no referent. A descriptive term may be significant, and understandable, as a description even though there are no test items in terms of which the term is defined.

In summary, the problem of conveying meaning with respect to what is classified in the taxonomy is complicated by unclear reference. In cases of classifying objects, terms used to refer to classes can be defined by referring to common properties which then become the defining characteristics of the classes. Ostensive definitions can be employed, in which cases the terms defined must have clear referents. For example in the case of animals, the total class is divided into vertebrates and invertebrates, vertebrates are divided into mammals, and non-mammals, non-mammals are divided into birds and non-birds, and birds are divided into reptiles and non-reptiles. Each class and sub-class can be unambiguously defined in terms of common properties. The problem of constructing a taxonomy is that of completely describing "that which is to be classified," then stating verbal definitions. The term which names a class or sub-class (definiendum) is defined in terms of descriptions which separate that class from all others. Language merely functions as the instrument in terms of which we conceptualize the nature and structure of that which is classified. We use language to impose meaningful order on the data of experience.

The problem of defining "educational objectives" would be considerably simpler if terms such as "teacher intentions", "expected student behavior", etc., referred to objects and events which could be ostensively defined, exhaustively described, and then verbally defined in terms of descriptions.

The authors of the taxonomy recognize this difficulty and organize their explication so that the meaning of what is classified emerges in the act of classifying. Type I, definitions are probably at first only empty abstractions, type II definitions function either as operational or as ostensive definitions. If they function operationally, the reader probably feels that he is not being told directly just what the term means, and if they function ostensively, meaning must be abstracted by forming a concept of common properties. Type III definitions do not define educational objectives directly. Each test item functions as an ostensive definition of the class characteristic of which it is an instance. Test items also function as operational definitions insofar as it is understood that what is defined (student behavior) is not classes of test items, but classes or types of responses to test items. The implied operational definitions are of the form. "X is the cognitive process which represents the way students are to think, feel or act, etc. when responding to a certain type of test item". Any class or sub-class name is a substitution instance of X. That is, X can be any name that is defined in type I definitions. Descriptions (type II definitions) could also be substituted for X in which case the definiens rather than the definiendum would be operationally defined.

In conclusion, the phrase "What is to be classified" implies reference in terms of which meaning can be conceptualized and understood. But since reference is at best indirect and conveyed either operationally or ostensively, the meaning of "what is to be classified" can only be progressively explicated in the transaction of interpreting the abstract categories for the raw data denoted by "teacher intentions", "expected student behavior", etc. Emphasis thus shifts from what is to be classified to the classification scheme,[30] from reference to meaning. And the meaning of what is to be classified becomes the meaning of the language used for classifying. Finally, the shift to meaning is complete when the terms which name and describe "what is classified" are classified. The guiding principles can then be introduced. Otherwise a criterion such as completeness would be absurd, because the taxonomy would not be complete until every instance of a teacher intention was counted and sorted.

The Meaning of Taxonomy

The central question to which this section will be addressed is that of deriving the specific taxonomy which was defined rather than one of the "almost infinite number" of possible taxonomies.[31] The preceding section consisted in an attempt to show that the meaning of "educational objectives" is explicated in the transaction of applying the abstracted terms of the taxonomy to the raw data of "teacher intentions". The conclusion was that reference is implied when types II and types III definitions function ostensively and operationally. However, implied reference is not as clear as in the case of physical object referents. In terms of teacher intentions, educational objectives may be those expected student behaviors which teachers intend and express. Or, in terms of test-items, goals in the cognitive domain may be those items whibh elicit a similar response if responded to correctly. The obvious difficulty here is that the number of elements which constitutes the classes to which reference is made is not fixed or determined. Hence, in order to avoid the confusing task of sorting and classifying an infinite set, emphasis must shift to the defining elements (descriptions, or type II definitions), and finally to names (terms which name the classes and sub-classes). Definition of the final abstractions then consists in the process of explicating meaning, beginning with ostensive and operational definitions (type III), then stating common characteristics. In this way, a term (*definiendum*)is defined which is meaningful in terms of common properties or predicates and which refers to expected behaviors as instances of the *definiens*. If the defined term is meaningful (significant) then the meaning which is attached to descriptions is transferred to the final terms.[32]

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[30] The next section will be devoted to examining how definitions function in explicating the "taxonomy". The classes and sub-classes which constitute the taxonomy are abstractions which must be developed from various referential contexts. Emphasis will be more on how abstractions function in explicating the meaning of terms which are related to other terms in a theoretical network, rather than on the states of affairs to which the abstractions refer.

[31] The use of this phrase (p.13) is evidence that the authors of the taxonomy were aware of problems in both meaning and reference. If classes were isomorphically related to their referents, there would apparently be only one possible taxonomy. On the other hand, if names were chosen and relations assigned without regard to referents, the range of possible choices would be infinite rather than "almost infinite".

The problem of what is meant or intended by the term "taxonomy" is well illustrated by the function which it serves in Handbook I. "Taxonomy" can function either as a name or as a description. If it functions as a name, the referent is the three handbooks: two published and one unpublished. In this case, the one word refers to the names, descriptions and illustrative examples which constitute the contents of the books in question. On the other hand, if the term functions as a description, the referents are those behaviors which are named, described and illustrated by the terms, descriptions and illustrative examples respectively.

In order to further clarify the problem of deriving an abstraction, as mapped out above, a typical example will be presented and discussed. Evaluation was chosen for this purpose since the lower levels will also be included (the taxonomy is hierarchical in structure). As emphasis shifts from reference to meaning a commentary will be given on problems involved in linking abstracted categories and meeting the criteria stated in the guiding principles. A more complete discussion will be presented after the introduction of additional terminology in the following chapter.

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[32] This is analogous to the problem of what is meant when reference is made to the set of natural numbers. The term 'natural numbers' cannot refer to a set of existent symbols, since the class is infinite. Therefore, meaning must be transferred to defining characteristics. That is the process of counting operationally defines the set of natural numbers.

An Illustrative Example

The explication of evaluation begins with a verbal definition (type I).

> Evaluation is defined as the making of judgments about the value, for some purpose, of ideas, works, solutions, methods, material, etc.[33]

The definiendum 'evaluation' is defined in terms of the definiens "the making of judgments, etc." The limitations of verbal definitions are immediately confronted. If the definiens were given as only "the making of judgments", then it might be interchangeable with the definiendum. But if the reader does not know the meaning of 'evaluation', he probably would know what is intended by "the making of judgments". The authors of the taxonomy attempt to avoid this circularity by introducing the phrases "for some purpose" and "of ideas, works, solutions, methods, materials, etc." However, in doing so, the rule for verbal definitions is violated. It may not be necessary to state what the purposes are (any purpose may be included). But in the last phrase, instances are cited of the kinds of situations which are the objects or contexts about which evaluations are made, the definiens given fails to meet the criteria for verbal definitions. The reader is expected to form a concept of the type or typs of situations which serve as contexts for evaluation; [34] i. e., the class of situations is ostensively defined. In other words, an ostensive definition has been introduced in the definiens of a verbal definition.

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[33] Bloom, op. cit., p. 185

The sentence following the one quoted above is an attempt to further explicate meaning.

> It (evaluation) involves the use of criteria as well as standards for appraising the extent to which particulars are accurate, effective, economical, or satisfying. The judgments may be either quantitative or qualitative, and the criteria may be either those determined by the student or those which are given to him.[35]

This explanation is probably intended to be a continuation of the type I definition which preceded it. That is, it is not a list of educational objectives (type II); nor is it a set of test items (type III). To say that evaluation involves such and such does not make clear whether "that which is involved" constitutes all of the defining characteristics of 'evaluation'. Presumably, if this were the case, the verb 'is' rather than 'involves' would have been used. Therefore, "that which is involved" must contain some defining characteristics of 'evaluation' but not necessarily all of them. Hence, the definition is of the form "cognitive processes which include these characteristics, and others like them, are called 'evaluation'. This statement may then function as either an operational or an ostensive definition. If a student successfully engages in tasks involving such and such, he is performing at the level of evaluation (operational). The examples (what is involved) also function as instances to which the term 'evaluation' is applied (ostensive).

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[34] It is not clear whether the examples given are each elements of one class, or whether each example is one instance of a separate class.

[35] Bloom, op. cit., p. 185

The last part of the quotation consists of two statements about the class of judgments. "The judgments may be either quantitative or qualitative, and the criteria may be either those determined by the student or those which are given to him". These statements refer to the entire class of judgments, but they do not specify defining characteristics. After the class is defined, it can be divided quantitatively and qualitatively. Also that there be criteria for making a judgement is important, but it is not important whether the teacher or the student specifies what it is. The class could also be divided many other ways (e.g., according to whether the judgment involves the judges, whether it is time bound, etc.).

Two kinds of criteria constitute the defining characteristics of two sub-classes within evaluation.

> One type of evaluation can be made largely on the basis of internal standards of criticism. Such internal standards are for the most part concerned with tests of the accuracy of the work as judged by consistency, logical accuracy, and the absence of internal flaws . . . A second type of evaluation may be based on the use of external standards or criteria derived from a consideration of the ends to be served and the appropriateness of specific means for achieving these ends.[36]

The criteria thus may or may not be an intrinsic part of the subject-matter about which the judgment is made. "Logical accuracy" apparently means clear and unambiguous use of terms or clarity of expression. "Consistency" means the avoidance of contradictions, etc. "Internal criteria" are therefore not subject-matter bound. External criteria are, however, closely related to the fields about which judgments are made. External standards are those which have been established for a particular field of inquiry. Evaluating a given work according to established standards is, therefore, largely a matter of working within the context of a given field. However, regardless of whether internal and external criteria are subject-matter bound, each is the name of a sub-class within evaluation.

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[36] *Ibid.*, pp. 186-87

"Judgments in terms of internal evidence" is defined as "evaluation of a communication from such evidence as logical accuracy, consistency, and other internal criteria".[37] and "judgment in terms of external criteria" is defined as "evaluation of material with reference to selected or remembered criteria".[38]

Each of these type I definitions is apparently of the _definiendum_ - _definiens_ form. However, an important distinction must be noted in the function of each for explicating meaning. The _definiens_ of "judgments in terms of interral evidence" contains a phrase which functions as an ostensive definition: "logical accuracy, consistency, and other criteria".

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[37] _Ibid._, p. 188

[38] _Ibid._, p. 190

That is, these two terms refer to instances of internal criteria and the reader must then abstract the meaning of "internal criteria" from what these two instances have in common.

In the case of external criteria, any criterion which is relevant to a given context would count as an instance. The only restrictions included in the definiens are therefore those of selection and remembering: an external criterion cannot be used if it is not available. The definiendum "judgment in terms of external criteria" is probably interchangeable with the definiens, "evaluation of material with reference to selected or remembered criteria". The only question is whether the definition is circular.

At this point, type I definitions have been discussed for the general class of evaluation and two sub-classes. In two cases, the definitions provided were of the definiendum-definiens, verbal form, but the definiens actually functioned as an ostensive definition. In the other case the definiens does not introduce instances of the denotation of the term. Hence, the type I definition of "judgment in terms of external criteria" is a verbal definition in both form and function.

Type II definitions (i. e., illustrative educational objectives) are given for the two sub-classes but not for the class itself.[39] Judgments in terms of internal evidence are illustrated by the following:

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[39] This raises the question of whether or not class membership is exhausted by members of the sub-classes. There is no reason to believe that the sum of sub-classes equals the total class. However, since no type II or type III definitions are given for the class separately, it must be assumed that the former is exhausted by the latter.

> . . . the ability to assess general probability of accuracy in reporting facts from the case given to exactness of statement, documentation, proof, etc.
> The ability to apply given criteria (based on internal standards) to the judgment of the work.
> The ability to indicate logical fallacies in arguments.[40]

The second sub-class of evaluation (external criteria) is illustrated by:

> The comparison of major theories, generalizations, and facts about particular cultures. Judging by external standards, the ability to compare a work with the highest known standards in its field -- especially with other works of recognized excellence.
>
> Skills in recognizing and weighing values involved in alternative courses of action. The ability to distinguish between technical terminology which adds precision to a test by permitting more appropriate definition of terms, and that which merely replaces a common name by an esoteric one.[41]

Few type III definitions (test-items) are presented in Handbook I. This may be due to the excessive space required to present lengthy items. However, another problem is that of scoring items which represent the highest level in the taxonomy. Scoring at the lower levels is comparatively simple. If a student selects the correct response on a knowledge level item, it is a simple matter to infer that he possessed the requisite knowledge. But the ability to make judgments using internal and external criteria is not so easily measured. Each of the five items quoted below represents a different aspect of the problem.[42]

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[40] Bloom, op. cit., p. 189

[41] Bloom, op. cit., p. 192

[42] These five items are the only ones given for judgments in terms of internal evidence.

Given a problem, determine a logical conclusion and judge the logical accuracy of statements in relation to the conclusion.

Social Security officials sometimes face perplexing problems in studying appeals for unemployment compensation. Some major-league baseball players in Ohio and Missouri decided in January that, because they had not played ball since the end of the season, they had a right to consider themselves unemployed. Although some of them were earning good salaries of $8,000 to $10,000 a year in baseball, they maintained that they were entitled to the benefits of the unemployment section of the Social Security Act.

Team owners urged these players not to apply for unemployment compensation. According to the owners' interpretation, the players were under contract all year around, although they worked and were paid only during the playing season. On the other hand, the state officials in Ohio and Missouri were inclined to agree with the players that they were entitled to benefit payments.

Directions: Examine the conclusions given below. Assuming that the paragraphs above give a fair statement of the problem, which one of the conclusions do you think is justified.

Conclusions

A. The players were entitled to the benefits of the unemployment section of the Social Security Act.
B. The players were not entitled to the benefits of the unemployment section of the Social Security Act.
C. More information is needed to decide whether or not the players were entitled to the benefits of the unemployment section of the Social Security Act.

Mark in column
A: Statements which explain why your conclusion is logical
B: Statements which do not explain why your conclusion is logical
C: Statements about which you are unable to decide.

Statements

1. The state officials are the ones who consider appeals for unemployment compensation, and their opinion carries more weight than the opinions of team owners.

2. The Social Security Act may or may not provide that a man who works and receives pay during only part of each year is unemployed during the remaining part of the year.
3. The players argued indirectly (if others receive compensation, why should't we?) and forgot that others needed the compensation more than they.
4. A changed definition may lead to a changed conclusion even though the argument from each definition is logical.
5. No one who earns $8,000 a year or more should get unemployment compensation.[43]

The following item represents judgments in terms of external criteria.

Given an end, recognize best of several means

Many people believe that it would be better if our states had more uniform divorce laws. It is recognized, however, that there are dangers in an attempt to achieve such uniformity. Which one of the following procedures would be most likely to avoid the greatest of tnese dangers?

A - An amendment to the U. S. Constitution is passed, which establishes the grounds for divorce to be recognized in all courts.

B - A federal law is passed which sets forth the maximum grounds which can be recognized by any state.

C - A commission appointed by the President works out standards for a divorce code and encourages all states to consider these standards in revising their laws.

D - A conference of state governors decides on a divorce code and each governor attempts to have it made the law of his state.

E - The U. S. Supreme Court establishes a uniform set of practices by ruling against all divorce laws which do not conform to its standard.[44]

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[43] Bloom, op. cit., pp. 196-97

[44] Ibid., p. 200

At this point, the problem of explicating meaning is that of using Type II and Type III definitions to add significance to the definiens of Type I definitions. If the meaning of each class and sub-class is established, (i. e., if the defining characteristics are significant for the reader), then additional educational objectives and test items can be specified. In other words, common characteristics are generated by the use of either ostensive or operational definitions, or both. These defining characteristics then become the definiens of the words which name the classes. Without type II and type III definitions, those of type I would be vacuous or empty of meaning. But to have the former without the latter would result in inefficiency and lack of economy in expression.

The specific problem, then, is that of determining how the defining characteristics of 'evaluation' and the two sub-classes are generated.

Again, since no type II and type III definitions are given for 'evaluation' itself, we must assume that the class is coextensive with the two sub-classes. Looking first at, 'judgments in terms of internal evidence' the initial problem is whether or not the defining characteristics of the class are generated from the three instances of educational objectives given. If each instance contributes meaning then similarities will define the genus and differences will distinguish what is not essential. What, then, is common and what is different between the first two instances of 'educational objectives'? According to the first example, 'judgment in terms of internal evidence' means to possess the

ability to predict "accuracy in reporting facts, from the care given to exactness of statements, documentation, proof, etc." And in the second case, the sub-class means "the ability to apply given criteria . . . to the judgment of the work." The first instance describes how the internal criterion is generated (noting care given to the exactness of statement, documentation, proof, etc.), and the ability defined is that of using the generated criterion to judge the work in question. The only difference between example 1 and example 2 is that in the latter case the criterion is given; someone else generates it internally from the work. Other than this, the first two instances are the same. But according to the definition of 'evaluation' "the criteria may be either those determined by the student or those which are given to him". Therefore, with respect to generating meaning, the first two examples are the same.

In the third case, the ability to indicate logical fallacies in arguments, the question arises as to why this is not an instance of judging in terms of external criteria. A fallacious argument is not a contextual matter. The type I definition of this sub-class included the term 'logical accuracy', but this ability seems to be described in the last example quoted under judgment in terms of external criteria. The inescapable conclusion is that the meaning of 'judging in terms of internal evidence', as ostensively and operationally defined by the

thr given educational objectives, is not clear. Since test-items perform the same function in the explication of meaning, an examination of these may help.

In the test situation presented, the student is expected to "judge the logical accuracy of statements in relation to the conclusion". No information is provided as to how the items are scored. But, since (c) is the only defensible conclusion, the five given statements must be assessed in terms of whether or not they support this particular conclusion. The statements are the criteria. That is, a conclusion is stated and the problem is that of giving reasons to support the selected conclusion. Statements 1 - 3 represent internal evidence or criteria, but statement 4, the one which best supports conclusion C, is not a matter of internal evidence. What statement 4 clearly says is that given a valid argument, changes in the premises will lead to a different conclusion, but this is the case in any argument regardless of content. Again, this criterion seems to be external rather than internal.

'Judgment in terms of internal evidence' probably means the ability to read a document and judge it in terms of such criteria as style, research evidence, accuracy, etc. Then, in the simplest sense, such a judgment would involve reading a newspaper editorial and proclaiming its narrowness in scope. Or a judgment could be sufficiently broad to include predictions about future cultural and social conditions based on an assessment of documents relating to present conditions.

The problem in assigning meaning to "internal evidence" or "standards", "consistency", "logical accuracy", and "the absence of flaws" is that each is not independent of the others. Consistency and logical accuracy are criteria for lack of internal flaws. Hence, the latter could constitute the defining characteristics of judgments in this sub-class, consistency and logical accuracy would then be instances of lack of internal flaws.

The meaning of this sub-class may therefore be divided into two parts: (1) the ability to examine a document and locate internal flaws, and (2) the ability to state reasons in support of the claim that a document contains internal flaws. That is, if criteria are not external they must be generated from given contexts. And to say that there are flaws is not sufficient; judgments must also specify why the alleged flaws actually function as flaws in the given context. The problem of assigning significance to 'judging in terms of internal evidence' is, therefore, that by definition the meaning of the manifested ability is a function of the particular context in question. If 'flaws' are defined across contexts, the criteria become external rather than internal. The problem of defining the class is that what is defined is actually a process in which the context may vary from situation to situation. What is central is the ability to judge rather than the criteria for judging or that about which the judgment is made. This sub-class must therefore be operationally defined. Consequently, insofar as type II definitions function operationally, they help in the explication of meaning, but appear to be circular. The second example is the case in point. "The ability to apply given criteria (based on internal standards) to the judgment of work" actually says that the sub-class is defined as the process of applying given standards, etc. Also, insofar as the first example functions operationally, it is of the same form, but the internal criteria are generated by the student rather than supplied to him. Both the first and last examples also function as ostensive definitions insofar as each describes a class or kind of activity. The ability to note exactness of statement, documentation, etc., describes an instance of how an internal criterion is generated, and the ability to assess general probability of accuracy in reporting facts is an example of a judgment made in accordance with the generated criterion. The last example may also function ostensively if "logical fallacy" is interpreted as contextually generated rather than as the name for a set of external criteria.

In principle the test item can also function ostensively. It is an instance of applying criteria supplied by someone else (the given statements). But since basically only one example is given, it is impossible to determine what two items have in common. The example can, however, function operationally. The sub-class means the process of selecting a conclusion (A, B or C) and then deciding which of the criteria (statements) support the conclusion and which do not.

To summarize, the type I definition of 'judgments in terms of internal evidence' is "Evaluation of a communication from such evidence as logical accuracy, consistency, and other internal criteria". The term logical accuracy is not significant unless it is interpreted for a particular context. If 'consistency' is not interpreted as a formal, external criterion (p. p), then it also becomes significant contextually. The sub-class then becomes operationally defined as the process of examining documents, determining internal flaws and judging why the instances cited are flaws. This definition may not be significant, but lack of significance is characteristic of operational definitions.

The problem of explicating what is intended by "judgments in terms of external criteria" is not as difficult. Since, by definition, the meaning of "external criteria" is not contextually bound, instances can be supplied. In the first sub-class, criteria had to remain unspecified. In the second sub-class (external criteria) ends are given and judgments consist in selecting the best means to these ends. Or the end may function as a standard in which case a comparison can be made, indicating how or in what way a given situation or state of affairs differs from the standard. Each of the test-items supplied illustrates means-ends relations. The items therefore function ostensively by pointing out or calling attention to a particular instance of means-ends problem. In the example quoted, given that it would be better to have more uniform divorce laws (the ends), whcih of the alternative means would best bring about the desired state of affairs? Another example given is an instance of "Given a means-ends relationship, judge its validity and support this judgment".[45] Thus, the meaning of 'ends' as external criteria, and 'relating means to ends' is defined ostensively by what is common to the abilities measured by the two test-items.

In a similar manner, educational objectives (type II definitions) provide significance for the definiens "Evaluation of material with reference to selected or remembered criteria". What is meant by 'judgments in terms of external criteria'? "The comparison of major theories, etc." "The ability to compare a work with the highest known standards in its field, etc." "Skills in recognizing and weighing values, etc." These statements function as operational definitions insofar as the reader equates the meaning of the terms to be defined with the processes described in each of these instances. Also, each functions as an ostensive definition if the reader attempts to abstract the defining characteristics from the examples presented. Instances of ends define the external criteria, and examples of relating means to ends define the act of judging. The difference between type II and type III definitions is, therefore, not in their respective functions, but in the level of generality within inquiry at which the function occurs. Type III definitions provide instances of a particular judgment in terms of external criteria. Meaning is generated by abstracting the defining characteristics of each type. In the case of type II definitions, each example is a description of a particular generic group of which a test-item represents an instance. To generalize across types or kinds, in abstracting the meaning of educational objectives, is thus to determine the defining characteristics of the class or sub-class; i. e., to supply significance for the definiens of type I definitions.

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[45] Ibid., p. 199

Ostensively, the meaning of 'judging in terms of external criteria' is what is common to the examples given: i. e., relating means to supplied ends. "Means" and "ends" then become significant in terms of the individual experience of the reader. Operationally, the meaning of the sub-class is the process of relating means to ends, where the emphasis is on what sort of activity takes place when an ability is manifested. That is, focus is on the solution to the problem rather than on the problem itself. In other words, if the sub-class is defined behaviorally, both the problematic situation and possible alternatives in interpreting and responding to it must be thoroughly specified. To explicate meaning ostensively, is to characterize the kinds of problems to which responses are expected. Operational definitions explicate meaning by presenting the kinds of responses which are expected in a given situation.

To summarize, the meaning of 'judgments in terms of external criteria' is explicated if the definiens "Evaluation of material with reference to selected or remembered criteria" has significance. "External criteria" may be interpreted as ends, as a standard to which a given state of affairs can be compared. Beyond this, meaning is a contextual matter. Thus, to further explicate the definiens requires the examination of possible contexts (type III definitions) and instances of possible types of contexts (type II definitions).

In summary, the meaning of the class 'evaluation' has been explicated if the two sub-classes are significant either as names or as descriptions. Probably the first sub-class, judging in terms of internal evidence, is less significant because the definiens presented is actually an operational definition. Otherwise, the criteria become external. The second sub-class, judgment in terms of external criteria, is more significant because the term 'external criteria' denotes ends or standards which do not have to be generated from given contexts. Specific ends can be defined ostensively and the meaning of the sub-class can also be presented operationally as the process of selecting and relating means to ends.

One final question remains: Is the meaning of 'evaluation' restricted to the meaning of the two sub-classes? If 'evaluation'means "the making of judgments", are there any alternatives to judging with respect to internal evidence and judging in terms of external criteria? Or is 'evaluation' possibly super-ordinate to the two sub-categories? These questions will be taken up later when the criteria of the guiding principles are imposed. But the meaning of 'taxonomy' cannot be fully explicated without at least inquiring whether or not the term 'taxonomy' refers to all possible educational ojbectives. If it does not, then meaning is not complete until those which are excluded are sorted and the taxonomy is amended to include them.

Similar discussions could be presented for other levels and sublevels. The question of whether a class is only as broad as its subclasses is particularly relevant in the case of comprehension. But aside from the problem of completeness, the 'taxonomy' in the simplest sense, refers to a set of six levels, each of which includes two or more sublevels. These names, in turn, refer to descriptions which constitute the defining criteria for classes of educational objectives. And, finally, each description provides the definition of a class of test-items which constitute the procedure for verifying the existence of the behavior referred to as the operationally defined construct. 'Taxonomy' means the significance which is attached to names in the taxonomy and 'taxonomy' refers to the behavior intended by these names. Thus, the question, "what does 'the taxonomy' mean"? becomes "Have necessary conditions been met for making 'the taxonomy' significant for X?" and since significance is a function of reference, X must think of the kinds of situations to which the term refers; i. e., meaning must be in terms of X's individual experience.

CHAPTER III

EXPLICATION OF THE TAXONOMY IN A CONTEXT OF MEANING-REFERENCE

In chapter II the focus was on the fundamental role of definitions in the explication of meaning. This was the first step in clarifying the extent to which meta-theoretical assumptions are made in deriving the taxonomy as opposed to a taxonomy. Definitions, themselves, are meta-theoretical, or as Brodbeck says, "about language".

> A definition, no matter how simple or complicated its form, is always a statement about the use of words. It asserts that one or more words may be used as an abbreviation for some combination of several others. Strictly speaking, therefore, definitions are about language rather than within language.[1]

However, the subject matter which is ordered by definitions or about which meaning is conveyed, is at the level of theory in a particular subject area. Therefore, definitions provide the connecting link between theory and meta-theory.

The purpose of Handbook I is to explicate the meaning of the taxonomy for the reader. The intent is to insure that meaning is conveyed rather than to focus on how this is done. This dissertation is an attempt to do the latter. The first step in doing this was to show how the three kinds of definition given in Handbook I (types I, II, and III) actually function as ostensive, operational and verbal definitions.

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[1] Brodbeck, op. cit., p. 47

The levels and sub-levels are then sorted and explicated using the three types of definition as criteria. Consequently, any assumptions which are incorporated into the definitions are automatically passed on to that which is ordered by the definitions (the taxonomy). The next step, then, is to make a meta-theoretical examination of definitions themselves.[2] That is, before any decision can be made in regard to whether the derived taxonomy is not merely one of several alternatives, the necessary conditions which have to be met in order to support such a claim must be ascertained. Then, if these conditions are fulfilled, the derivation is successful.[3]

In chapter II the words 'meaning', 'reference' and 'significance' were used as non-technical terms in ordinary language. In the present chapter, however, these terms must be used in a precise way. Both

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[2] This examination will be concerned with ostensive, operational and verbal definition in order to be consistent with the terminology used in the literature. Type I, type II, and type III definitions could be used if the differences were merely in names. But, since type II and type III definitions can function either as ostensive or as operational definitions, there would be ambiguity.

[3] Only necessary conditions are possible here because no guarantee can be given that the sufficient conditions for universal acceptance can be met. For example, it is necessary to assume (for completeness) that any educational objective after analysis will correspond to one level or sub-level in the taxonomy. But is this true by definition (because the authors say that it is) or because of the nature of what is classified? Even if the latter is the case, it is still defining elements (meaning) which is classified. Otherwise, the authors face the impossible problem of classifying an infinite set (that to which the term refers).

'meaning' and 'reference' are general terms which can be subdivided into more specific and more technical parts. Quine makes the distinction between meaning and reference quite clear and gives examples from each sub-area.

> When the cleavage between meaning and reference is properly heeded, the problems of what is loosely called semantics become separated into two provinces so fundamentally distinct as not to deserve a joint appellation at all. They may be called the theory of meaning and the theory of reference. The main concepts in the theory of meaning, apart from meaning itself, are synonymy (or sameness of meaning), significance (or possession of meaning), and analyticity (or truth by virtue of meaning)---The main concepts in the theory of reference are naming, truth (and) denotation.[4]

It will be unnecessary here to deal extensively with each of the sub-areas which Quine names. Within reference the concern will be principally with denotation and, to a lesser extent, with naming and truth. In the context of meaning, significance will be discussed, but connotation will be added, as an analog to deno tatinn. Also, descriptions will be treated as a sub-area of meaning, since the emphasis will be on verbal relations rather than on that to which the descriptions refer. The following diagram shows how the six areas are ordered.

A. Meaning
   1. connotation
   2. significance
   3. description

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[4] Quine, W. V., From a Logical Point of View, p. 130

B. Reference

1. denotation
2. truth
3. names

Each of these terms will be defined as briefly as possible and then further explicated in the discussion of each type of definition.[5]

Connotation - Denotation

The connotation-denotation distinction is used here instead of intension-extension because of the ambiguity sometimes associated with the latter pair.[6]

> The connotation of a word is the characteristic or set of characteristics which are such that these characteristics determine the objects to which the word can be correctly applied, and which are therefore sufficient to mark out these objects from other objects . . . The denotation of a word is the object, or set of object, to which the word rightly applies. For example, the denotation of "triangle" will be all triangles. Thus, "triangle" connotes all the objects that have

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[5] A question which will be taken up in the next chapter is emerging here. Briefly, it is that of the meaning of 'meaning' or the definition of 'definition'. Subsequently, an attempt will be made to show that envoking the theory of logical types avoids circularity. But at this point, it is only necessary to note that definitions of 'definition' are subject to the same limitations as those of 'knowledge', 'comprehension', etc.

[6] For example, to say that the intention of a word is all that we intend by it, suggests an overlapping of psychology and logic. Stebbing presents a clear summary of the different meanings of 'intention' which were originally distinguished by J. M. Keynes. See Stebbing, L. S., A Modern Introduction to Logic, p. 28

these characteristics.[7]

Cohen and Nagel give a briefer but similar definition.

> A term may be viewed in two ways, either as a class of objects, or as a set of attributes or characteristics which determine the objects. The first phase or aspect is called the denotation or extension of the term, while the second is called the connotation or intension. Thus the extension of the term "philosopher" is "Socrates", "Plato", "Thales", and the like; its intension is "lover of wisdom", "intelligent" and so on.[8]

When someone gives an ostensive definition, he is pointing to one instance of the term's dentotation. When type II and type III definitions function ostensively they, therefore, present to the reader, one educational objective or one test item which constitutes one member of the class or sub-class of which it is an instance. This will be discussed in greater detail after the other pairs of terms are defined.

Names - Descriptions

'Names' and 'descriptions' are terms frequently used in every day discourse and are familiar to most people. Probably few persons would confuse a name with a description. Nevertheless, it is necessary to insure that the technical, meta-linguistic use of these terms is clarified before focusing on limitations and assumptions involved in the three kinds of definition.

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[7] Ibid., p. 28

[8] Cohen, Morris R., and Nagel, Ernest, An Introduction to Logic and Scientific Method, p. 31.

For the sake of simplicity, logically proper names will be contrasted to descriptions and then class names such as 'knowledge', 'comprehension', etc. will be contrasted to proper names. Stebbing stresses that a logically proper name has no connotation.

> . . . it should now be clear that a logically proper name, such as "this", has no connotation; its function is purely demonstrative. An ordinary proper name, however, since it contains a descriptive element is not purely demonstrative; it certainly has subjective intension. If connotation be not taken as equivalent to definition, an ordinary proper name may be said to have restricted connotation, the restriction being to the context within which the proper name is being used. It is obvious that descriptive phrases are connotative.[9]

A logically proper name is thus a word or symbol with stands "for a referent without ascribing any characteristic to it".[10] The adverb 'logically' is used to refer to those terms which have no connotation as opposed to those proper names which may contain "a descriptive element"; i. e., have some connotation. For example, a proper name such as "Franklin" denotes all those persons who are so named, but, conceivably, it may be possible to state defining characteristics of these individuals (connotation of Franklin) and, hence, to distinguish these people from all others.

Similarly, class names have connotation as well as denotation, insofar as we can state the defining characteristics of a class. But there is still an element of naming involved, since there are no rules which determine just which word corresponds to the defining characteristics. In other words, after the defining characteristics of a class such as interpretation have been abstracted from instances given in type II and type III definitions, there are no rules which state that this word and no other must be used to name the class. Yet, since, 'interpretation' has widely accepted connotations, we do not have the freedom to choose any symbol whatever to name this sub-level.[11] It is not important here that there be any logically proper names; what, however, is important is that the function of naming is clear. The same point can be made by noting that non-connotative, logically proper names would only refer, since there is no connotation, there is no overlapping with meaning.[12]

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[9] Stebbing, L. S., *A Modern Introduction to Logic*, p. 30.

[10] *Ibid.*, p. 31

Descriptions are the meaning analogs of names. But as Russell has shown, whereas it is necessary to have a referent in order to name, a description can be meaningful whether or not there is an object to which the description refers.[13]

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[11] For example, if the connotations of 'John' were fixed, parents would err in designating their child by this name if he did not possess the necessary defining characteristics.

[12] If no distinction is made between meaning and reference, this non-overlapping may not be fully appreciated. Quine, borrowing from Russell and Frege, presents a good argument for preserving this distinction. See, Quine, op. cit., p. 21

[13] When the authors of the taxonomy state that there is an almost infinite number of possible taxonomies, what they apparently mean is that names are not descriptively determined (that the connotations are not fixed), and there is sufficient freedom in assigning names to permit an almost infinite number of products.

Quine stresses the word-word relation of meaning as opposed to the word-object relation of reference by saying that "meaning is what essence becomes when it is divorced from the object of reference and wedded to the word."[14]

Stebbing gives a very clear account of descriptions by contrasting "a descriptive phrase" to a "demonstrative symbol".

> . . . a descriptive phrase ascribes characteristics. These characteristics may, or may not, belong to some object. If there is any object to which these characteristics belong, then the descriptive phrase applies to that object, or objects. For example, if there is no individual who is both honest and a politician, then "honest politician" is a descriptive phrase that describes nothing . . . The reason why we can thus understand and use descriptive phrases, even when there is nothing to which such phrases apply, is that their meaning is independent of the context within which on any given occasion they may occur . . . Descriptive phrases are thus to be sharply distinguished from demonstrative symbols . . . A demonstrative symbol may be regarded as a logically proper name, since it symbolizes an object with which there is acquaintance, and which is thus immediately given.[15]

At this point, a distinction should be made between descriptions and connotations, on the one hand, and names and denotation, on the other. Again, to look for a rigid, formal difference would probably be misleading. The distinction is rather one of function. In the process of explicating meaning, going from operational and ostensive definitions to verbal, definiendum-definiens statements, emphasis is first on reference. Thus, as meaning is abstracted, emphasis shifts from the instances or elements which constitute a given class to the word which names the class. When a term is defined ostensively, each example presented is one instance of the term's denotation. When common characteristics are abstracted from two or more examples, simple descriptions are generated. But these descriptions are not sufficiently inclusive or exhaustive to be called the defining characteristics of the word. When the process of abstraction is more advanced, it is possible to at least attempt to state the defining characteristics. These, then, are the connotations, C. I. Lewis is probably more explicit on this point than other writers.

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[14] Quine, op. cit., p. 22

[15] Stebbing, A Modern Introduction to Logic, p. 25

> Formally considered, the intension of a term is to be identified with the conjunction of all other terms which must be applicable to anything to which the given term would be correctly applicable.[16]

This definition may appear circular; that is, it would seem that we must first know the meaning of a term before we can say what the connotations are. In other words, we cannot know the connotations unless we know the connotations. However, Lewis is discussing definiendum-definiens type definitions in which circularity is only avoided if the meaning of the definiens is abstracted from instances and then incorporated into one very general statement or a conjunction of words and phrases. Therefore, 'connotation' and 'description' may in many cases be used interchangeably. But the latter term refers to common properties abstracted early in the synthetic process, while 'connotations' refers to the defining characteristics which when conjoined constitute the meaning of the term (definiens) and are more general or abstract than simple common properties. After all connotations are generated and expressed in the definiens, the exact term defined may still not be uniquely determined. That is, the name given to a class which has been so defined is not determined by the conjoined connotations. But neither is there almost complete freedom as when on assigns a proper name. Thus, the relation naming other than in the case of logically proper names, is functional rather than formal.

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[16] Lewis, C. I., An Analysis of Knowledge and Valuation p. 39

The distinction between names and denotation is probably clearer than that between description and connotation, but there is at least one problem about which confusion must be avoided. For C. I. Lewis, the distinction is made in terms of existence: what is denoted must exist, while names may be used to refer to things which may or may not exist.

> . . . if the non-existent can be named, then naming is not denoting; . . . things are, or would be namable by a term but which do not in fact exist, are not included in the denotation.[17]

However, when existence is not made the defining criterion for naming, the distinction between names and descriptions become less clear. Returning to Stebbing's example of honest politician," . . . if there is no individual who is both honest and a politician, the "honest politician" is a descriptive phrase that describes nothing".[18] Thus, if names are also used to refer to non-existent things, then the name actually functions as a description. But for the purpose of this dissertation, names will be subsumed under reference and used in the functional sense of logically proper names. Names, then, as demonstrative symbols, are purely referential. And, as such, it is absurd to seek other members of the named class (in which case the name would function descriptively). Denotation, on the other hand, emphasizes similarities within a class of objects, things, or events. Thus, when a name functions as a logically proper name, reference is made to individuals or classes taken as units. On the other hand, a term denotes all members of a class. There may be only one member, but the expectations are that there will be many.

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[17] Ibid., p. 39

In summary, the differences between descriptions and connotations, on the one hand, and names and denotation, on the other, are differences of degree or emphasis rather than of type or kind. What is important here is that such a distinction be made so that the function of each can be noted in deriving the taxonomy. Early in the abstractive process, we seek common properties across two or more instances of raw data and are therefore describing. After we abstract the defining characteristics (connotations) of a homogeneous class, then we can give it a name. Names and denotation are both in the context of reference and differ in emphasis rather than type: the focus in naming being on

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[18] Stebbing, A Modern Introduction to Logic, p. 25

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individuals while denotation emphasizes class membership collectively.

Significance - Truth

The third pair of meta-theoretical terms to be discussed is 'significance' and 'truth'. Significance is an aspect of meaning, and truth is best understood within the context of reference. However, these terms are not analogs of each other as was the case with connotation-denotation and description-naming.

Quine gives a simple definition of significance as the "possession of meaning".[19] Lewis is slightly more explicit, "The signification of a term is that property in things the possession of which indicates that the term correctly applies, and the absense of which indicates that it does not apply".[20] Brodbeck gives an example of a term which she says is signficant, "if it appears in a law connecting it with other concepts".[21] She continues:

> A good example is the concept "cephalic index". This concept can be quite adequately defined operationally in terms of the ratio of the width of the person's head to its length, with the method of measurement precisely stated . . . Since "cepholic index" does not enter into laws, the concept is useless for predicting or explaining behavior. It lacks "meaning" in the sense of significance.[22]

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[19] Quine, op. cit., p. 131

[20] Lewis, op. cit., p. 39

[21] Brodbeck, op. cit., p. 57

[22] Ibid., p. 57

This example should not be construed to mean that significance is discovered in laws. Nor is significance a matter of individual understanding. This point is made quite clear by Bechtoldt who, after quoting Brodbeck and Bergmann, says, " . . . a defined concept may be without significance but significant concepts must be defined".[23] In other words, a definition can be formulated, but there is no guarantee that it will be meaningful (i. e., significant) because there may be no connections with other terms, laws, research, etc. Thus, even though it is absurd to say that a term may be meaningful even if no person knows its meaning, still significance is not merely a matter of individuals being familiar with a term and using it. Rather, significance may be said to be a precondition which must be met before a term can be used in discourse.

Truth, on the other hand, is a matter of reference. To begin with an example from the taxonomy, Is translation a sub-class of comprehension? This question could be more clearly phrased as, Is it true that translation is a sub-class of comprehension? In the first question, it is not clear what would count as support for either a negative or an affirmative response. If someone says "yes", is he referring to the classification scheme in Handbook I? Or is he referring to the educational objectives which constitute the raw data or subject-matter for Handbook I? In other words, why is the structure or hierarchy true?

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[23] Bechtoldt, op. cit., p. 623

because the authors of the Taxonomy say that it is, or is it because educational objectives referred to in the Taxonomy are so ordered? If the arrangement presented in Handbook I is not arbitrary, it must be empirically grounded, which necessitates that we consult the raw data of teacher intentions, etc. as the criteria for truth.

The problem of the definition of 'definition' is particularly evident in the case of truth. Russell notes that Tarski was responsible for showing the meta-linguistic nature of 'true' and 'false'.

> Tarski . . . has shown that the words 'true' and 'false', as applied to the sentences of a given language, always require another language, of higher order, for their adequate definition.[24]

Logicians sometimes say that truth is a primitive property of propositions which is transmitted from the premises to the conclusion in a valid argument. But just what is meant by 'truth' is left unanalyzed and undefined. In fact, Russell classes 'true' and 'false' with other logical words and states that they cannot occur within the object language.

> What is certain is that the object language does not contain the words 'true' and 'false' or logical words such as 'not', 'or', 'some', and 'all.'[25]

But, since the word 'true' does occur in less formal discourse, and since truth is a property we expect to find associated with propositions, even in theoretical contexts, some attempt must be made here to define it.[26]

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[24] Russell, Bertrand, An Inquiry into Meaning and Truth, p. 59

[25] Ibid., p. 73

We may not seek truth in such basic assumptions as "all behavior is motivated" and "the universe is helio-centric", yet we expect verifiable consequences. Otherwise there would be no good reason to make such an assumption. If someone says "A is intelligent" or "it is true that A is intelligent", we would have a set of behavioral expectations for the person referred to as A. Thus, for the purposes of this dissertation, 'truth' may be operationally defined as "the process by which a hypothetically described state of affairs is verified. The definition focuses on the predictive quality of truth and may ignore prior evidence which was taken into consideration when the claim was made. This point, however, merely emphasizes the temporal dimension in inquiry. If certain consequences can be predicted from *X is the case*, then presumably these, or similar, consequences must have occurred in the past. Russell points out the empirical nature of truth.

> . . . a proposition is 'all men are mortal', assuming it true, derives its truth from 'A is mortal', 'B is mortal', etc. And

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[26] Quine points out that paradoxes must occur when terms within the theory of reference are treated as meaningful. "These paradoxes seem to show that the most characteristic terms of the theory of references, namely 'true' and 'true of' and naming (or 'specifying') must be banned from language as meaningless on pain of contradiction". See Quine, op. cit., p. 134

> each of these derives its truth from such propositions as 'A grows cold', 'B grows cold', etc.[27]

The meaning of 'truth' is thus more complex than those other topics within the theory of reference which have been discussed (naming and denotation). This is because in the latter two cases, the relations involve the mere existence of things, objects, events, etc., while in the case of truth, the relation involves properties of propositions. In other words, in order to understand the meaning of 'name' and 'denotation', it is only necessary to understand that certain objects, things, events, etc. exist and are denoted or named. But to understand that a proposition is true entails an understanding that things, objects, or events stand in a particular relation to each other. And this relation is in some way mirrored, pictured, or shown in the proposition about objects, things, etc. Thus, truth, or at least empirical truth, is established by observations rather than by reflection, and, for this reason, is within the domain of reference rather than of meaning.

The Function of Ostensive Definition: Empirical Grounding

The foregoing discussion has been an attempt to provide a working vocabulary, within the basic meaning-reference distinction, for a more meta-theoretical explication of the taxonomy. Within the conceptual framework now available, the following questions can be asked.

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[27] Russell, *op. cit.*, p. 273

1. Do terms used in the classification scheme function as logically proper names? i. e., could any noise or conjunction of letters be used? Or, is it the case that class and sub-class words, which are as a matter of fact used, are entailed by the nature of what is classified?

2. What is the nature of the empirical grounding on which the taxonomy is based? There are three corollaries to this question:

    (a) Is the taxonomy defined according to logical criteria (meaning) with empirical reference a matter of accident?

    (b) Do the divisions in the taxonomy merely reflect those which exist in the data to be classified?

    (c) Do the raw data provide the implicit lines of division which are made explicit, distinct, and unambiguous, etc. by the logically defined classification scheme?

3. Do the data to be classified occur as actual existents to which the terms refer or do the terms merely connote hypothetical instances which could exist but which may or may not have occurred?

Other more tangential questions could be raised, but those given are central for the purpose of this dissertation. That is, clearly, defining the taxonomy is not merely a matter of meaning. Type I or verbal definitions are not formulated with respect to logical criteria alone nor, on the other hand, do categories emerge from the data in such a way that no logical rules are applicable. The abstract terms defined by type I definitions involve both reference (raw empirical data) and meaning (a conceptual organization of terms which is meaningful apart from supportive empirical data). Therefore, the remainder of this chapter will be devoted to a meta-theoretical discussion of what conditions must be met in order to have an abstracted taxonomy which derives meaning from empirical data, and, yet which gives meaning to the same uninterpreted data.

The logical order of learning the meaning of a term begins with instances which function as ostensive definitions and terminates when the learner can give a significant (for him)verbal definition. In order for the type I definitions in Handbook I to be significant, the learner, or reader, must go through this process. A type III definition presents one instance of a kind or class of educational objectives. Type II definitions also function ostensively by presenting a description of a sub-class as one instance of a more generic class. Empirical significance abstracted from instances is then stated verbally as the definiens of a type I definition. This process must be such that no empirical significance introduced in ostensive definitions is lost. But, on the other hand, the abstractive process must not permit instances to be included which are not members of the defined class. That is, if the taxonomy is to meet logical criteria, then any two persons should not be able to generate contrary defining characteristics.

In the following discussion it must be remembered that the meaning-reference distinction is a logical or conceptual device the function of which is to provide greater clarity. Within the framework provided, the explication is discursive rather than ostensively defined as "what the reader of Handbook I does". In addition, if certain necessary conditions are not met, this should also be clear.

Question (3) will be discussed first, and within this analysis, 2-C will emerge as the correct alternate. But a further discussion will be required to show why this is so. The question can be rephrased as follows: when educational objectives are defined ostensively, are instances given of the term's denotation? That is, when we learn the meaning of words like 'dog', we first observe instances to which the term applies; then we abstract the defining characteristics, so that ultimately we can tell someone what a dog is without having to point to physical object dogs and say "that is one". In a similar way, the data on which the taxonomy is based must be empirical instances - the term's denotation. If this requirement is not met, then the empirical foundation of the taxonomy is at least weakened and at most lost. In other words, if a clear line is not drawn between connotation and denotation, or between meaning and reference, the hierarchy of classes and sub-classes is fundamentally a conceptual arrangement of terms, the truth of which is settled by reflection rather than by verification.

Specifically, then, are the raw data on which the taxonomy is based actual instances denoted by the class and sub-class terms? The answer is no. Cognitive levels and sub-levels are defined in terms of student behavior but not actual student behavior. What is actual is teacher intentions. However, the terms in the taxonomy refer to student behavior. What is classified then is student behavior intended by teachers.

Thus, since teachers are not bound by actual performance, they can set their expectations as high as they wish.

The argument can be better understood by examining a specific example. Suppose that a physical education teacher formulates a set of objectives for the broad jump. No reasonable teacher would state absurd goals such as fifty yards. Nor would anyone stipulate extremely low expectations. But how are reasonable expectations defined? The answer is that they are defined empirically. What students can do - not what teachers may expect, no matter how reasonable the expectations may be.

In general, probably all educators stipulate reasonable goals. But the line between what is reasonable and what is not reasonable is not determined by what anyone can meaningfully assert. The line is defined by actual student behavior. To start with intended student behavior is to begin on the meaning side of the logical fence and, for this reason, must admit a degree, however small, of prescriptiveness.

The last point requires some amplifications; 'what can be meaningfully described' does not have to exist. Teachers can present descriptions of student behavior even though no student has ever performed in accordance with the stipulated expectations. Thus, just as Russell can say, "The present king of France is bald" is meaningful, even though there is no present king of France, teachers can meaningfully descirbe expectations either as test items (type III) or as kinds of test items (type II), and it is an accidental matter whether or not any student can meet the defined goals. The prescriptive element therefore enters when teachers state goals which, implicitly, should be met.

Before proceeding further with the analysis, an additional point must be considered. It has been said that type II and type III definitions function as ostensive definitions, and the function of ostensive definitions is to present instances of a term's denotation. Thus, if raw data which are classified in the taxonomy are not instances which the class and sub-class terms denote, then can it still be said that type II and type III definitions present meaning in the manner of ostensive definitions? In order to be as clear as possible on this point, certain aspects of a subsequent discussion will be introduced here. It might be argued that class and sub-class terms are hypothetical constructs and therefore can only be defined operationally. Significance then is just the set of operations in terms of which a term is defined and no more. The point to be supported hereis that if this were the case, verbal or type I definitions would be vacuous. It will be argued that meaning (significance) is implicit in instances which are presented. Significance is then abstracted from contexts, restricted by operational definitions, and finally formulated as the defining characteristics in a type I definition.

When it is said that type II and type III definitions function ostensively, what is meant is that with respect to generating meaning from given instances, the process is the same regardless of whether the data are in the form of actual existing instances or verbal descriptions of actual existing instances. Verbal descriptions are therefore the meaning analogs of empirical events, but the requirement of actual occurence is not imposed. Thus, when the reader is confronted with either an actual instance or a verbal instance, the problem of generating defining characteristics is the same - what do two or more instances have in common? It matters little with respect to generating meaning whether persons who are learning the classification scheme observe instances qua material events or instances qua simple verbal descriptions of empirical events. But the important difference is that when someone describes an educational objective, there is no guarantee that the corresponding event in the empirical world can or does exist.

To summarize, if raw data, i. e., descriptions, are at a low level of abstraction and "mirror" empirical events, then the presentation of a verbal example serves the same functinn as the presentation of an actual example. Each functions ostensively in as much as it is implicit in each case that "this" (what the terms refer to) is an example of what the term means. The reader or learner must then determine what he thinks is the most noteworthy aspect of two or more instances and posit this as the defining characteristic. Of course, at the early stages, what is abstracted is probably merely a common property which will be subsequently combined with other properties, reduced and formulated as a type I definition.

The thesis here supported is therefore that each test item is a description which functions as an ostensive definition of a member of a sub-class. Test items, are actual, but what is defined is a response to the items, and this is not actual but expected. When the reader then focuses on what is common to two or more test items qua expected response, he has begun to form a concept of a type or kind of behavior. Type II definitions also function ostensively, but in this case, a description presents a kind of educational objective. For example "Skill in predicting continuation of trends" and "The ability to differentiate value judgments from predictions of consequences" are examples of extrapolation. Each functions as an ostensive definition with respect to the class itself, and test items perform the same function with respect to the sub-classes. Type II and type III definitions are therefore similar in that each by itself is an instance of behavior or a kind of behavior from which defining characteristics of classes and sub-classes respectively are generated. However, there are differences with respect to levels of generality (degree) and with respect to whole-part (kind). That is, classes, are more genral than members within the classes, and, with the exception of the most primitive sub-classes, classes and sub-classes can be divided into more elementary parts.

In the language of meta-theory a term or name may or may not have any denotation. If we begin the process of abstraction by referring to empirical data as objects or things, then our terms necessarily have denotation. But since the terms in the taxonomy are defined within the domain of meaning, the possibility cannot be overlooked that some terms may have zero denotation. Probably all terms do have denotation insofar as at least some students should be able to meet our expectations; however, when we define behavior in terms of teacher intentions, we automatically include the possibility of zero denotation.[28]

The Function of Operational Definition: Limitations and Boundary Conditions

In the following discussion, meaning which is abstracted or generated from instances which function as ostensive definitions will be more specifically called empirical significance. The adjective, empirical, is, of course, not applicable if some terms have zero denotation. But in order to proceed with the explication, it will be assumed that meaning which is generated from given instances is empirically significant. The problem, then, at this juncture is to concentrate on how meaning is retained, reduced, and then formulated as the definiens of type I definitions.

Probably the clearest way to put the problem is to ask what restraints are placed on the abstractive process in order to insure that meaning is bounded. That is, since limits are not set empirically, what restrictions are placed on the process of abstraction or generalization in order to preclude the possibility of stating defining characteristics which would be so broad as to permit behavior not appropriate to the class in question?

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[28] In the chapter on meaning and language, Hullfish and Smith give some examples of terms which have zero denotation. See Hullfish and Smith, op. cit., p. 141

The restrictive function is performed by operational definitions. An operational definition " . . . identifies the meaning of a concept not with its referent, but with the operations one must perform if one wants to test the presence of a referent".[29] The referents of terms in the taxonomy are those behaviors which are "present" if a test item elicits a correct response. In the case of type II definitions, fewer restrictions are imposed. If a student can respond correctly in a situation which requires "the ability to differentiate value judgments from predictions of consequences", then he has performed at the cognitive level of extrapolation. Bechtoldt noted that "terms like 'hostility', 'agression', 'psychopathic deviate','intelligence', and 'mechanical ability' have some referents and some rules for their use".[30] The rules in the taxonomy are not explicitly stated, formed rules. Rather, each type II and type III definition functions as an operational definition or rule for applying the class and sub-class terms.[31] If a student performs "correctly" under certain sets of conditions, he has demonstrated an ability to perform at a particular level. If other rules are followed in other contexts, he has performed at other levels and sub-levels. As was the case with ostensive definition, differences between type II and type III definitions are mainly matters of degree. Many restrictions are implied in the statement of one test item, while the specification of certain kinds of abilities in type II definition results in grouping all behavior of a single type together and distinguishing this from all other behavior. The correct response to a test item then functions as a rule or operational definition. If a student responds correctly to the specific conditions defined by a test item, and if a correct response is made on all members of a set which are homogeneous with respect to possession of some property, then the student is said to have responded at the cognitive level defined by type III and type II definitions respectively. From a wrong response, such as an incorrect choice on a multiple choice item, the inference is made that the student does not know the rule. It is usually not possible to say that, in the case of a wrong response, another set of rules is being followed, i.e., that, for example, choice (a) would involve analysis and choice (b) would be evidence of synthesis. This is analgous to saying that if a player makes an incorrect "play" in a game of poker, then he is playing by the rules for bridge.

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[29] See footnote 15, p. 33

[30] See footnote 16, p. 33

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[31] The dual function of type II and type III definition may be more easily understood if we consider a more formal (explicated) situation such as a game of chess. It may be said that to be able to correctly play the game of chess is just to know the rules and nothing more. (In general, to know the meaning of a word means to know the rules for using the word, as evidenced by continued correct use). This is necessary but not sufficient for two reasons: (1) If this were all that is involved, then a properly programmed computer would be able to play the game, and man would only approximate a correct game, and (2) to make such an equation would result in overlooking the distinction between knowing the rules and applying the rules. In the operational sense, Handbook I is more of a set of rules for classifying behavior than a classification scheme. This is evidenced by the applicability of the question: Have test items been correctly classified? i. e., have the rules been properly applied?

Rules thus may be said to impose boundary or limiting conditions within a given context or problematic situation. But just as knowing rules is not sufficient for playing a game, operational definitions are not sufficient for defining the levels and sub-levels in the taxonomy. That is, the real test of skill in a game like chess is not knowing how and when particular pieces are permitted to move according to the rules, but rather the ability to apply the rules in a given problematic situation. The same point can be made in a less formal setting. We include operational definition when we define a kind of cognitive behavior as the ability to make correct responses to any or all of a set of test items. The common properties which collectively define the class are thus commonalities within the written items (ostensive definitions) and similarities or even identities in responses (operational definitions). In cases such as this, presumably there is no one who knows or can formulate all the rules (each response is an instance of the rule). But the test constructor at least knows the procedure used in writing items. If a person whose ability is being tested, knows the rule, he may make a correct response, however, the strongest evidence of a given ability is the application of the rule in a wide variety of situation.

In summary, we might say that in the case of intelligence testing, operational definitions are implicit strategies or rules to be followed across situations or contexts. If this were not so, test constructors would be equally as handicapped as test takers. This is also the case in respect to those who attempt to explain behavior (i. e., the users of the taxonomy and those whose behavior is to be explained).

Meaningful Verbal Definitions: Retained and Transmitted Empirical Significance

The next stop in the explication is to discuss how meaning, or empirical significance abstracted from instances which function as ostensive definitions, is transmitted to the definiens of type I definitions. It should be emphasized that the process is not formalized; otherwise, the explication here attempted would be preempted. When type II and type III definitions function as ostensive and as operational definitions, a contextual instance is presented. The reader must then abstract meaning (as empirical significance) from each instance, and he must infer the rules, which when interpreted in a given empirical setting constitute the instance. The process is analgous to that of observing a player follow the rules in a more structured situation. When we learn the meaning of 'chess', we learn what is relavant in terms of what rules are being followed. For example, it is necessary to have a particular kind of board, rather special appearing pieces, two players, etc. Other things are not relevant: it is not important whether the pieces are made of wood, ivory, etc., the color is not determined, etc. That is, the content is defined implicitly. Anything not present is presumed to be not relevant. The empirical significance of 'chess' is therefore what is relevant in a given instance and what is common to two or more such instances.

Similarly, when we learn the meaning of 'cognitive behavior' by observing instances, we abstract empirical significance (what is relevant) from the examples which we encounter. However, we can make only a limited number of transformations on the elements which constitute a problematic situation; i.e., limitations are implicit as rules to be observed if a response is to count as an instance of the kind of cognitive behavior in question. Rules and that to which rules apply are both necessary components in the process of defining levels in the taxonomy; in the process of abstracting and preserving meaning to be attached to categories in a meaningful, self-contained hierarchy. Significance must be attached to terms which denote (here describe) empirical events.

Presumably an infinite number of transformations could be performed on the given elements. But operational definitions restrict or limit the number of acceptable operations.[32]

The dual function of type II and type III definitions has been discussed as if they were separate and distinct. Actually each is a concomitant aspect of the explicative process. Also the generation of meaning is not a uniform, one way process from less to more abstraction. Each hypothetically meaningful abstraction functions as mediator between all past observed instances and each newly encountered one. The abstract terms acquire meaning from past instances and gives meaning to future ones. This is the explicative process.

The problem for the reader of Handbook I is to use the examples presented by the authors as stepping stones in the abstractive process.[33] And the obligation to do this is the reader's. The authors can provide verbal definitions in which connotations are significant in terms of the descriptive words used in the illustrative examples, etc.; that is the authors can give type I definitions which meet certain meta-theoretical requirements, but they cannot generate empirical significance for any individual reader. There is nothing the authors can do to prevent someone from reading type I definitions first and then generating meaning in terms of what he would intend if he were formulating the definitions. In other words, the classification scheme, qua type I definitions, is a hierarchy of abstract terms. But what the terms connote is empirically significant if and only if the reader has engaged in a process of controlled inquiry called 'explication'. If readers should choose to regard the abstract classification scheme as sui generis, apart from supportive data, then meaning would not necessarily be empirical significance; it would be anything from the reader's experience which is coherent or conceptually tenable. Truth would not necessarily be empirical truth, terms could be merely names given to the classes and sub-classes.[34] If such is the case, continuity between the data on the one hand, and the abstract hierarchy of terms, on the other, would be lost. A bifurcation would have been made and there would be apparent grounds for the claim that the authors "do not consistently subscribe to an operational behaviorism", and that the taxonomy has "metaphysical overtones".[35] If the reader chooses to assign metaphysical meaning to the terms used in Handbook I, i.e., if cognitive processes are given non-empirical, non-verifiable consequences, then he has not followed the explication implicitly presented by the authors.

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[32] In a much simplier context the process of abstraction is illustrated by the learning of "addition". What does 'addition' mean? The term acquires empirical significance in the contexts where in the operation is defined: blocks, fruit, etc. But just given empirical objects, any number of operations could be performed, viz stacking, placing in rows, etc. The observer learns from the context what transformations are included. Finally the learner becomes able to deal with "pure" abstractions on a symbolic level, but the abstracted symbols still retain their empirical significance in as much as the person who has learned them knows that they apply to blocks, fruit and similar objects. The process is carried out in accordance with certain boundary or limiting conditions (distributive, commutative, and associative laws.) Hence, if the term 'addition' is defined at the abstract level, the *definiens*, as what the term connotes, contains very general, abstract terms which are meaningful both as empirical significance, or contexts to which the terms apply, and in terms of specified transformations as operations which can be carried out within the ostensively defined contextual range.

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[33] There is ample evidence throughout the Handbook that the authors expect the class and sub-class terms to become significant for individual readers through the kind of explication discussed here. An entire chapter (3) is devoted to practice exercises in classification. The "correct" answers are supplied (p. 59) as a check of the reader's understanding.

Mnay years ago Dewey wrote in *Essays in Experimental Logic*

> . . . that instrumental necessity has led to a metaphysical hypostatizing of meaning into essences or subsistencies having some sort of mysterious being apart from qualitative things and changes is a source of regret; it is hardly an occasion for surprise.[36]

"Hypostatizing" occurs when terms are associated with non-empirical referents. Meanings become fixed, and as such, constitute the limits which bound inquiry rather than function as instruments which aid the process of inquiry and which themselves become reconstructed as a result of the constant reconstruction of experience. The taxonomy is defined within the domain of meaning beginning with actual descriptions of expected behavior and terminating with verbally defined classes and sub-classes. 'Explication' is the process whereby meaning progressively emerges through an interaction between words and their referents within the reflective activity of an inquiring individual. Explication is therefore antithetical to hypostasis. This chapter has been an attempt to say *how* the explicative process is carried out. The reader of the taxonomy must see *that* it is carried out.

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[34] That is, terms which designate levels and sub-levels would be more like logically proper names which *only refer* as opposed to words which are descriptively determined in which case an error would be made if other words were used.

[35] See footnote 4, p. 3.

[36] Dewey, John, *Essays in Experimental Logic*, pp. 46-47.

CHAPTER IV

THE TAXONOMY IN RELATION TO CRITERIA STATED IN THE GUIDING PRINCIPLES

The purpose of the present chapter is to examine the abstracted taxonomy in terms of the guiding principles which Bloom *et al* state as conditions to be met. These principles are:

> First, . . . the major distinctions between classes should reflect, in large part, the distinctions teachers make among student behaviors . . . A second principle is that the taxonomy should be logically developed and internally consistent ... A third principle is that the taxonomy should be consistent with our present understanding of psychological phenomena . . . A fourth principle is that the classification should be a purely descriptive scheme in which every type of educational goal can be represented in a relatively neutral fashion . . . Another way of saying this is that any objective which describes an intended behavior should be classifiable in the system.[1]

The First Principle: Correspondence with Empirical Data

The first principle states that major distinctions between classes should be grounded in what teachers say and do. The phrase "in large part" indicates that the logical distinctions made in the completed taxonomy may be sharper than in the supportive data. The second principle is that of logical classification. The terms should be so defined that the categories are ordered according to specific criteria and are free of contradictions. The third principle is that of external consistency with respect to psychological research. And finally the fourth and last principle is that the taxonomy should be purely descriptive and complete.

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[1] Bloom, *op. cit.*, pp. 13-14.

Very little can be added to what has already been said regarding the first principle. Before defining the classes and sub-classes in the manner which was, as a matter of fact, done, the authors apparently examined and reflected upon the distinctions which teachers make as evidenced by available data. The explication in which the reader of Handbook I engages then should confirm that major taxonomic distinctions (logical distinctions at the level of meaning as opposed to reference) correspond to those in the illustrative data supplied in the Handbook (assuming, of course, that the samples are both randomly selected and representative). If the reader finds that some data do not correspond to distinctions made in the taxonomy, then he must attempt to reconcile the differences. If differences seem irreconcilable then correspondence with reality (truth) must be questioned. The reader should not make the mistake of treating the taxonomy as a set of rigid criteria which cannot be falsified.[2] Regardless of the level of abstraction, it must be remembered that the taxonomy is empirically grounded and, therefore, can be falsified by instances which are counter to the defined structure. Hence, to assess the success or failure of the Taxonomy on the criterion of principle one is an on-going matter which will require additional research and careful analysis.

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[2] For example, if the proposition "all crows are black" is treated as analytically true, (functions analytically), the occurrence of a white crow would not falsify the statement. The person for whom this proposition functions analytically would merely say, "This bird is not a crow, it is not black".

For a thorough discussion of the analytic function in inquiry, see Hullfish and Smith, op. cit., chapter 6.

The Second Principle: Logical Development and Internal Consistency

Logical sub-division can only take place after divisions and subdivisions within the data have emerged and are apparent. Otherwise, constructing the taxonomy is a purely logical matter. The second principle is further clarified by the statement" . . . each category should permit logical sub-divisions which can be clearly defined and further sub-divided to the extent that appears necessary and useful".[3] Stebbing defines logical division as the " . . . process of splitting a class into its constituent sub-classes . . .",[4] and notes that there are rules for accomplishing this.

The principles regulating a logical division are usually summed up in the following rules:

1. There must be only one fundamentum divisions (defining characteristic) at each step.
2. The division must be exhaustive.
3. The successive steps of the division (if there be more than one) must proceed by gradual stages.

From Rule 1, there follows the corollary that the classes must be mutually exclusive. Violation of this rule results in the fallacy of cross division or overlapping classes. Rule 2 secures that no class be omitted, so that the sum of the sub-classes shall equal the whole class divided or classified. Rule 3 secures that each stage of the sub-division should be in accordance with the original fundmentum divisionis.[5]

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[3] Bloom, op. cit., p. 14.

[4] Stebbing, A Modern Introduction to Logic, p. 434.

It is clear that the taxonomy does not completely meet these three logical requirements and an attempt to assess the extent to which it does, would require a separate dissertation. However, the usefulness of a set of criteria is merely in setting standards. It is expected that the standards will be met only in degrees. Therefore, the rules should not be interpreted as inviolate laws which must be met if the taxonomy, is not to be regarded as a failure, but rather as a set of guidelines which, if followed, will yield a product that can be efficaciously used in ordering experience. For example, Rule 1 requires that there be one defining characteristic of interpretation, translation, and extrepolation which makes each of these categories a sub-class of comprehension. But rule 1 also stipulates that there cannot be more than one defining characteristic. That is, whatever the reason is for including interpretation as a sub-class of comprehension, the same reason explains why translation and extrapolation are also included. And all other reasons are irrelevant for this classification.

Rule 3 requires that successive divisions be in accordance with the fundamentum divisionis. In the most general sense, this means that a principle of division appropriate to the cognitive domain be used throughout. Criteria appropriate to either the affective domain or the psycho-motor domain would not be relevant for Handbook I.

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[5] Ibid., p. 435

Rule 2, however, requires a more detailed analysis. Why should the sum of the sub-classes equal the whole class? Again, the "Rule" should not be construed as a law which governs the process of scientific inquiry. But there are reasons why an attempt should be made to follow the rules as closely as possible.

As an example, consider knowledge, the lowest level in the taxonomy. The sub-classes are (1.10) Knowledge of Specifics, (1.20) Knowledge of Ways and Means of Dealing with Specifics, (1.30) Knowledge of Universals and Abstractions in a Field (See Appendix B). Do these three sub-classes and their respective divisions exhaust the class of knowledge? It is clear that knowledge in the sense of "the Knowledge explosion" is never complete. But when we use phrases like this, what do we mean by 'knowledge'? There appear to be three possibilities. (1) We are referring in a very general way to the entire cognitive domain, (2) We mean that there is no limit to the number of times the class knowledge can be sub-divided, or (3) The sub-classes generated by any given division are not always complete. This would mean that there may be classes other than those given at the level of the first sub-division, and there may be additional classes at each subsequent level.

Alternative (1) would be merely a matter of noting the technical sense in which the term is used in the taxonomy. The second alternative is sometimes expressed as," we know more and more about less and less". This interpretation may account for the incompleteness of knowledge. However, to carry out the process of sub-division indefinitely would blur the distinction between kinds of knowledge and that about which we have knowledge. Thus, if by alternative (2) we mean that about which we can have knowledge is unlimited, we are merely pointing out the obvious.

The possibility cannot, however, be ignored that we may intend the third alternative. What evidence do we have that all the kinds of knowledge which represent a given sub-class are determined and represented in Handbook I? Or, what is a more primitive question, are there any advantages to having exhaustive sub-division at a given level? These questions will be discussed concurrently.

If we consider the question of exhaustive sub-classes with respect to the class knowledge, the question becomes whether we must always speak of knowledge of such and such or can we speak of knowledge itself? In other words, if it is possible that there are sub-classes other than the three given, then the meaning of knowledge would have to be broad enough to permit their derivation.

Perhaps the problem will be clearer if we examine an example in which what the term refers to is clearer. If we classify the animals, would we take as our first sub-class all of those animals which have a given property or characteristic (fundamentum divisionis) and say that the residual animals (those left over after the classification has been carried out) are not kinds of animals, but are instances of "animalness" itself? Clearly, this would be grossly unscientific. Or in the case of triangles, we would not expect someone to say that there are isosceles, scalene, and equilateral triangles, and there are also just triangles themselves. And, if someone made such a claim, we would say that he was mistaken. However, if we expect the category 'knowledge' to transcend the three sub-classes of knowledge, we are making the same kind of mistake.[6]

Thus, when we read the paragraph describing the class itself, we must understand that this is a more general description than that provided for the sub-classes collectively, but the referents are the sub-classes and nothing more. Similarly, the description given for each sub-class derives meaning from its sub-divisions.

The problem can be further put into focus by examining the second level in the taxonomy (comprehension). This level is less complex insofar as there are only three sub-levels and no further sub-divisions. The central question is then, does the meaning of 'translation', 'interpretation', and 'extrapolation' exhaust the meaning of 'comprehension'? Or, in more behavioristic terms, is there some way to test for comprehension other than in terms of the three sub-levels? There are no test-items for the class itself. Nor are there any illustrative educational objectives.[7] But the fact that verbal definitions are given for the class itself apart from the sub-classes raises the question of whether or not, how, or in what way, the meaning of the class can be broader than the collective meanings of the sub-classes. This question will be given further consideration under the discussion of the fourth guiding principle.

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[6] All the sub-classes and divisions of sub-classes are titled knowledge of something.

Returning to the question of why it is desireable to comply with Rule 3, the answer is that the Rule functions as an axiom implying that it is desirable to have clarity and economy of expression which promotes communability; i. e., compliance with Rule 3 is conforming to an axiom basic to all scientific inquiry.

The problem of exhaustive sub-classes can be rephrased and summarized in the language of this dissertation. Meaning is broader than reference, as connotation is broader than denotation, in thesense that if we know the meaning of a word, we can recognize instances to which it applies. However, this distinction is between the meaning of a term and that to which the term applies, and, in the taxonomy, between the class and sub-class terms, on the one hand, and what is classified, on the other. The kinds of student behaviors qua reference are infinite, but the kinds of student behaviors qua meaning are finite. The logical criterion of exhaustive sub-classes applies in a context of meaning. Therefore, we are dealing with the defining characteristics of classes rather than with instances of classes. Hence, a class *qua* meaning (connotations) is broader than a class *qua* reference (denotation) in the sense that additional members of the class can be identified. But the meaning of a class is no broader than the combined meaning of the sub-classes, albeit, the level of abstraction is higher. That is, the definiens of a class is more general, more abstract, and briefer than the conjunction of descriptions or predicates about the class, but there is no residue of meaning left over. The meaning of the sub-classes conjoined is the meaning of the class itself.[8]

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[7] Neither are there any type II or type III definitions for the other classes per se.

The Third Principle: External Consistency With Respect to Psychological Research

In general, the logical criterion of external consistency is merely the requirement of internal consistency in a broader context: the totality of scientific research. The specific statement that the taxonomy must be consistent with psychological research is a reaffirmation of the close connection between education and psychology. Some indication of a possible relation to psychological data is indicated by the statement that ". . . since the taxonomy deals only with educationally intended behavior, if falls considerably short of being a classification scheme for all psychological phenomena".[9] Thus, just as in the case of complying with principle one, short of having all psychological data collected and ordered, we cannot assert that consistency with". . . our present understanding of psychological phenomena" has been established. Hence, principle three must be regarded as an operational rule which serves as a continuous criterion for relating the hypothetico-theoretical taxonomy to the emerging matrix of psychological research. And if inconsistencies are revealed, then an attempt at reconciliation through reconstruction must be made. The alternative is a loss in efficacy with respect to explanation and communication.

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[8] It should be noted that since terms acquire empirical significance from instances to which the terms refer and retain it in the progressively abstract process of formulating definitions, if class terms have residual meaning beyond that of the sub-classes, any such meaning could not be empirical significance.

The Fourth Principle: Descriptiveness and Completeness

Prima facie, the requirement that the taxonomy be descriptive opposes description to prescription - 'Description', in this sense, is a sy nonym for 'empirical'. However, in the more technical sense in which the term is used in this dissertation, a description mirrors, or is isomorphically related to, an observable property. We cannot say "observed property" because the taxonomy does not begin with what has been actually observed; expected student behavior is observable behavior. At the lowest level of abstraction, actual, existing descriptions of expected student behavior are the meaning analogs of actual existing instances of student behavior. Thus, although descriptions are empirically based (function as ostensive definitions in the taxonomy) a class of described events is not co-extensive with a class of actual events which have occurred, but with a class of possible events whose actual occurrence is contingent upon given circumstances. Hence, for every instance of a possible educational objective, there is a verbal description which corresponds to it. Descriptions, then, are the raw, unsorted data which are classified. The *kinds* of educational goals which will emerge and constitute the divisions of the taxonomy are manifested, but unclassified. It is for this reason that descriptions can function referentially. Although, strictly speaking, a description is verbal and is within the nexus of symbolic expressions rather than the domain of physical or material events.

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[9] Bloom, op. cit., p. 14.

There is thus a more formal requirement for descriptions. To say that a work of science is descriptive rather than prescriptive is probably necessary but not sufficient to rule out what some have called 'metaphysics'. The authors of the taxonomy, hence, must meet the requirements of descriptiveness insofar as it deals with descriptions of empirical events rather than with intuitions, reflections, or something else. But the empirical chain of abstract meaning still has one weak link in it insofar as descriptions qua reference are not confined to actual events, but only to episodes which are possible.

The completeness part of principle four requires more than just an explanation. In order to support the contention that the classification scheme is complete, at least two axioms have to be formulated and stipulated.

Several aspects of the completeness criterion were discussed under the second principle, viz exhaustive sub-classes. The following discussion will be more general, and will attempt to show that exhaustive sub-classes is necessary, but not sufficient for completeness.

This requirement is stated very clearly in Handbook I, ". . . any objective which describes an intended behavior should be classifiable in this system".[10] But what guarantee could we possibly have that when someone states a new objective, it will be similar in type, or kind, to what is provided for in the classification scheme? For example, we should be able to classify books according to color, shape, or size, and know that we have provided for all possibilities. The lines of division might be arbitrarily stated, but once defined, there can be no ambiguity. A book must be in one category or another. But what guarantee could we have that a classification according to subject-matter is complete? If we have a classification scheme worked out, there is no reason to preclude the possibility that someone will write a book which is within two or more of our defined classes. If this happened, of which class would it be a member? One way to cope with the problem would be to say that according to our set of defining characteristics, there may be ambiguity, but we could set up a different set of defining characteristics in which the ambiguous case would be clearly in one class or another. This, however, does not solve the problem. If we used other defining characteristics, we would merely generate one of the other "almost infinite" number of possible alternatives". Thus, in order to derive the taxonomy which is complete, we do not have the freedom of stipulating other dedining characteristics. If we cannot have alternative possibilities, other taxonomies must either be reducible to the one generated, or merely be structured using other terms which have the same empirical significance. In other words, since meaning is generated from descriptions about empirical events, presumably the abstracted taxonomy cannot be greater than, or different from, the descriptive data. And if the future is like the past, expected educational objectives will be classifiable.

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[10] Ibid., p. 14.

The matter is, however, not so simple. Returning to the problem of ambiguity according to one set of defining characteristics, a well-known example is whether the whale is a fish or a mammal. According to one set of defining characteristics it is a fish. But according to another set, it is a mammal. Is the solution the addition of another category to the classification scheme? That is, the new class would connote only one member: those creatures we call by the name 'whale'. A corresponding example in the taxonomy is whether there is a class of educational objectives which includes behavior common to two sub-classes say interpretation and translation? Or to two classes, e.g., analysis and synthesis? Clearly, the solution is not to generate another class, because there would be no reason to prevent the continuation of such a process, in which case, the taxonomy could not be complete. The only sense in which the taxonomy can be complete is in terms of the defining characteristics in accordance with which the taxonomy is derived. In order to discuss how some set of defining characteristics may be developed, it is necessary to re-examine "what is classified".

"What is classified" refers to actual descriptions of intended behavior. The cognitive content of the phrase (i.e., What the reader thinks when he examines the phrase), is not actual student performance, but actual written or verbal objectives. The raw data to be classified, then, are actual teacher intentions which refer to possible student behaviors. However, it is not educational objectives qua actual teacher intentions which are classified, because if so, the classification scheme would only have to be sufficiently broad to encompass what teachers have and do state as their goals. Nor, can 'completeness' mean that the infinite set of possible student behaviors is classifiable. Clearly, then, the set of defining characteristics in terms of which the classification is carried out, must be finite. The central question is how can a finite set of defining characteristics be generated from an infinite set of raw data?

In the discussion of exhaustive sub-classes (principle two) it was pointed out that connotations, or defining characteristics, are broader than and explain instances to which a term applies. Yet the former must be finite, while the latter can be infinite. If we impose the condition of actual existence on instances of the term, the relation is between connotation and denotation. Stebbing uses the terms 'intension' and 'extension' and points out that one is inversely related to the other.

> The interpretation of 'extension'. . . has been held to involve what has been called 'the inverse variation of extension and intension'. What this means can best be made clear by means of an example. If we consider the classes: squares, rectangles, parallelograms, quadrilateral figures, we see that they fulfill the following conditions: (1_ each class (taken in order from first to last) includes fewer individual numbers than the succeeding classes (since, e.g., all squares are rectangles, but not all rectangles are squares): (2) each class has a greater number of characteristics than the succeeding class. Thus, for example, if we know that a given plane figure is a square we know a greater number of characteristics with regard to it than if we only knew that it was a rectangle.[11]

The connotations, or defining characteristics, of 'square' are greater in number than are those for 'quadrilateral figure'. That is, more restrictions are placed on "that which is a square" than are placed on "that which is a quadrilateral figure". The latter is "any four-sided, plane figure". While the former is "a four sided plane figure with all four sides equal and all four angles equal." But there are more quadrilaterals than squares.

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[11] Stebbing, A Modern Introduction to Logic, p. 29.

This relation would also hold between the connotations of terms in the taxonomy and instances to which the terms apply, where 'instances' are not quantitatively bound by the requirement of actual existence. The problem of abstracting meaning from a set which is not determined is a matter of reduction in meaning. Similarly, the reduction of meaning from a set of descriptions or predicates to a one statement definiens is a matter of conveying the same meaning in fewer words. The inverse relation between connotation and denotation, therefore, also holds within the hierarchy of meaning.

Thus, in order to meet the completeness criterion, we must assume that the meaning of a term is the conjunction of all predicates or descriptions which apply to instances of the term. That is, if we assume that the next instance would add to the meaning of a term, then meaning in a classification scheme which is empirically generated, would never be fixed.

The reductionist assumption is not unique to any abstract system. Whitehead and Russell had to state such an axiom in Principia Mathematia which they called "the axiom of reducibility".

> The axiom of reducibility is the assumption that, given any function $\phi \hat{x}$, there is a formally equivalent predicative function, i.e., there is a predicative function which is true when $\phi \hat{x}$ is true and false when $\phi \hat{x}$ is false". [12]

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[12] Whitehead, A. N., and Russell, Bertrand, Principia Mathematia, p. 56.

And, two pages later,

> The axiom of reducibility is equivalent to the assumption that 'any combination or disjunction of predicates is equivalent to a single predicate'.[13]

In the context of the taxonomy, this axiom functions descriptively as a statement that meaning is reducible from a set of conjoined (or disjoined) descriptions to one predicative function (the definiens), and, in the formal sense the axiom of reducibility is a necessary condition which must be stipulated at each level and sub-level, if a claim for completeness is made.

The axiom of reducibility is both a quantitative and a qualitative statement; the addition or subtraction of data will not affect the meaning of a class or sub-class, either quantitatively or qualitatively. This concern, however, is with each level separately. In other words, meaning is not a function of the number of instances from which it is abstracted. Nor is the scope of meaning affected by the addition or subtraction of instances; i.e., subsequent data will not require the division of a class into sub-classes which are different in type from each other.

The latter statement, however, raises the question of adequacy in regard to the present hierarchial structure in the cognitive domain. If additional data do not necessitate changes in the structure, when or how is a decision made that the presented taxonomy is complete and no modifications are necessary? No discussion of the logical problems of proving completeness can be given here, but it will be shown that an additional assumption is necessary.

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[13] Ibid., p. 58. Note that the word 'disjunction' is used in this quotation instead of 'conjunction'. This does not make any difference; the point is that any single or individual predicate is not essential to the defining characteristics.

The rather obvious assumption of representative data must, of course, be made. But given that all possible kinds of educational objectives are sampled and represented in the taxonomy structure, is it possible to have an example which is not classifiable in the present hierarchy? This is approximately equivalent to whether there are examples like the whale, which do not raise questions about the scope of the taxonomy, but about the internal structure. Such an example then would not be classifiable in the scheme, as presently structured, but would be evidence of "other taxonomies".[14]

Consider what might count as such an example. To function at one level in the taxonomy is a necessary condition for performance at the next higher level. We cannot expect behavior representing the level of comprehension unless a student first has knowledge. Similarly, a learner cannot perform at the level of application without first understanding a theory or mode of interpretation in terms of which an application is carried out. However, is it conceptually tenable that the order could be reversed? Are there cognitive goals which would be instances of a different arrangement than those that are given?

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[14] In a restricted sense, this question is the same as that of mutual exclusion. But it will be shown that an instance which appears to be in two overlapping classes actually represents a more complicated problem.

Prima facie, the levels appear to be so arranged that to expect behavior at the higher levels, in a given subject-area, prior to behavior at the lower levels would be like having the whole without the parts. For example, we cannot construct (synthesize) without first having defined the discrete units which go into the construction (analysis). However, if we examine specific cases, the matter is not so simple.

Assuming that we do intend, as a matter of choice, to teach methods of organization, reflective or critical thinking, etc., in a setting of formal education, we must design lesson plans in such a way as to present the specifics of these areas of inquiry. Thus, in a course in formal logic, we might begin by presenting rules of inference.[15] Presumably, subsequent lesson plans would consist in building on the simples which are provided first. Then, when the student can interrelate the elements of the subject, he comprehends the material. And, if he can apply the interrelated abstraction which he has "learned", he has demonstrated an ability to perform at the first three levels in the taxonomy. But the central question in teaching abstract or formal subject matter is whether knowledge of specifics functions at the same level as knowledge of principles and generalizations. If Joe knows that the cat is on the mat, he knows a specific fact. Or, if he knows that 2 + 2 = 4, he may be said to have learned a simple fact. But if he is told that it is a fact that when we have a statement of the form "If p, then q" and when p is given, we can infer q, he may ask "why?". Similarly, if the teacher tells Johnny, "Any two bodies in terrestrial space attract each other with a force which varies directly as the mass of the bodies and inversely as the square of the distances between them, "and Johnny leaves the room muttering "any two bodies . . .", we may say that Johnny has knowledge of a principle or generalization. But in the language of this dissertation, this "knowledge" will not have empirical significance. Or, according to the arrangement in the taxonomy, Johnny may not know what the principle means before understanding occurs at the next level.

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[15] Whether these are members of the sub-class knowledge of specifics, or at a higher level within knowledge, e.g., knowledge of principles and generalization, will remain to be seen.

Returning to the example of teaching critical or reflective thinking, it may be argued that this is merely a method of teaching rather than a subject to be taught. That is, any subject can be dryly presented, or introduced in a dynamic, problems centered way. This argument, however, assumes that reflective thinking is non-discursive, that it can be shown, but not taught. To teach a course in logic on the basis of these assumptions would involve teaching principles and generalizations as de facto truths that must be accepted without question, rather than as efficacious tools to be used in inquiry because they promote communication and explanation. Thus, for example, the law of transitivity (If A is greater than B, and if B is greater than C, then A is greater than C) would be presented not as a law to be committed to memory, but as a basic relation which functions as an axiom in accordance with which we reason.

The subject-matter of abstract courses could be presented according to the order in the taxonomy. Students would accept, as facts, specifics within the subject matter, as well as principles and generalization. As elements are related, understanding would be expected, then application, etc. But when abstract or formal material like mathematics is taught using the methods of reflective thinking, elements are not presented as meaningless principles to be learned at the knowledge level. Rather emphasis is on the "why" of each point. In such cases, understanding would precede, or be concomitant with, knowledge, but "understanding" is with respect to discourse from which the principles or rules were abstracted. And knowledge is in respect to the formal system considered as a self-contained unit apart from other dimensions of inquiry. Similarly, when we do not treat formal systems as self-contained but as statements abstracted from and which represent, other areas of inquiry, then test items at the level of application, qua the formal system, would presuppose an ability to perform at the higher levels with respect to any particular field of content. For example, if we teach a course called "methods of analysis in problem solving", and present a unit on hypothesis formation, then success on test items at the level of application in the taxonomy hierarchy *qua* the order of the material in the course would be predicated on success at analysis and synthesis *qua* the problem area about which hypotheses are made. Thus, *applications* of logic in problems of democracy might entail *analyzing* a problem in democratic institutions and *synthesizing* an hypothesis which could be defended in the context of political science.

Another example which is counter to the order presented in the taxonomy is the following. When the reader of the taxonomy is first presented with verbal definitions for each level and sub-level, it is not presupposed that the definitions are significant. Thus at this point, to test at levels above knowledge would be purposeless. If our intention is to teach the taxonomy as a logically abstract classification scheme, we could present type I definitions and expect students to have a *knowledge* of the taxonomy in terms of *principles and generalization.* And we could design a test to determine if our expectations are met. Does a student know that analysis precedes synthesis, etc.? If he has knowledge of this order because he has read Handbook I, then a test would be designed and interpreted on the basis that his knowledge is not preceded by cognitive behavior at the higher levels. But if the student knows that analysis precedes synthesis because he has confirmed this relation in some problem area, then knowledge might be preceded by cognitive behavior at *any* of the higher levels.

In general, when we read Handbook I and understand or confirm that the taxonomy is true (that the classification scheme corresponds to the facts and data of our experience) can we, indeed must we, if the taxonomy is complete, classify our cognitive goals within the taxonomy structure? The answer is self-evident. Since the taxonomy is a classification scheme for all educational objectives, what we are classifying (our cognitive behavior as we read Handbook I) must be classifiable within the scheme which we are classifying.

The problem is not new. What is involved is the logical problem of self-reference. If the taxonomy is complete with respect to all possible educational goals, then it must include itself, as an educational goal. That is, there is no reason to prevent someone from attempting the classification of all educational goals as their educational goal (as the authors of the taxonomy, as a matter of fact did).[16]

Russell recognized the problem which he discusses as a "philosophical aspect" of Principia Mathematica.

> Thinking along the lines which had hitherto seemed adequate, it seemed to me that a class sometimes is, and sometimes is not, a member of itself. The class of teaspoons, for example is not

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[16] Many examples of self reference have been given in the literature. A rather well known one is that of the town barber who shaved everyone in town who did not shave himself. The question is whether or not the town barber shaved himself. If he, as a citizen, did shave himself, then he, as the town barber, had to shave himself. Thus, if he shaved himself then he did not shave himself and conversely.

> another teaspoon, but the class of things that are not teaspoons, is one of the things that are not teaspoons. There seemed to be instances which are not negative; for example, the class of all classes in a class. The application of Cantor's argument led me to consider the classes that are not members of themselves; and these, it seemed must form a class. I asked myself whether this class is a member of itself or not. If it is a member of itself, it must possess the defining property of a class, which is not to be a member of itself. If it is not a member of itself, it must not possess the defining property of the class, and therefore must be a member of itself. Thus, each alternative leads to its opposite and there is a contradiction.[17]

Russell then introduced his well known axiom or theory of types in order to avoid this paradox.

> We shall have to distinguish between propositions that refer to some totality of propositions and propositions that do not. Those that refer to some totality of propositions can never be members of that totality. We may define first order propositions as those referring to no totality of propositions; second order propositions, as those referring to totalities of first order propositions; and so on *ad infinitum*.[18]

To make this kind of distinction may appear to be leaning on a logical crutch; to be merely pointing out the source of confusion rather than correcting it. Russell admits that the theory of types is probably not the best solution to the paradoxes, but by making distinctions within the domain of reference, we at least avoid conceptual confusion, which is more central in the taxonomy.[19]

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[17] Russell, Bertrand, *My Philosophical Development*, pp. 75-76.

[18] *Ibid*, p. 82.

[19] The discussion here is intended to be informal and explicative. A formal proof of completeness, or lack of, for the taxonomy would necessitate more structuring of the classification scheme. See Godel, Kurt, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, Translated J. B. Meltzer, Oliver and Boyd, London, 1962.

Thus, the problem of order in the taxonomy hierarchy requires the proviso that intended behavior must refer to a particular area of subject matter. Otherwise cognitive performance at a given level may presuppose behavior at a higher level in the taxonomy. And this would contradict the order defined in Handbook I. However, instances of this type probably could only occur when one area of discourse is about another area, or when we have, implicit, levels of discourse in which statements at one level refer to statements at another level rather than to objects and things experienced directly.

We confront the problem in a more obvious way if we attempt to classify learning of the taxonomy itself, as an educational objective, within the classification scheme. The whole appears to be equated to a part. The paradox is avoided, however, if we make a distinction between educational objectives which refer to a given area of content, on the one hand, and those which refer to educational objectives about areas of content, on the other. The terms in the taxonomy refer to total classes: all possible behavior of a certain type at a given level. Thus, when we state the learning of the taxonomy (memorizing type I definitions) as an educational goal to be classified at the knowledge level, we may be equating the whole to a part. The specification that objectives which refer to classes of student behavior are different in type from those which refer to objectives which, in turn, refer to classes, etc. avoids confusion, in general, and eliminates the generation of counter instances, in particular.

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary: Theoretical and Meta-theoretical Aspects

The specific problem to which this dissertation was directed was that of explicating meaning in Handbook I of Bloom's Taxonomy of Educational Objectives. This consisted in both theoretical and meta-theoretical analyses of " what is to be classified" and the "classification scheme". Specifically, an attempt was made to show that, if the taxonomy is to have an empirical base, the process of explicating meaning from the raw data of teacher intentions must be such that the definiens of verbal definitions contain empirical significance. The criteria according to which the abstractive process was explicated were those determined by three kinds of definition. Each type served a necessary, albeit not sufficient, function in the generation of a meaningful hierarchy of classes and subclasses.[1] The raw data of actual descriptions of behavior functioned as ostensive definitions of each class. Descriptions of educational goals are called type II definitions and sample test items are referred to as type III definitions. The relation between an educational objective, on the one hand, and test items which collectively define an objective, on the other, is therefore that of a class to members within the class.

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[1] No claim can be made for sufficiency since there can be no guarantee that the classification scheme will be meaningful for any particular individual.

Type II and type III definitions can function as either operational or ostensive definitions or both. Each test time presents an ostensive definition of that to which a term refers. And each type II definition presents one instance of a kind of educational goal. Both types also function as operational definitions; many transformations may be possible in a given empirical context, but only those which are accepted in a given situation are relevant to the meaning which is to be associated with a particular term.

The class and sub-class terms in the taxonomy refer to responses on test items and to abilities to perform correctly in certain situations and under specified conditions. When a test item is given as one instance to which a class or sub-class term refers, the empirical context is defined. For example, in a multiple choice item, what is relevant to the meaning of a term is implicit in the situation presented. But only certain transformations are accepted as evidence of the cognitive ability in question. If a student does not respond correctly, it is inferred that he has not functioned at the level which is being defined. Thus, what is empirically and contextually relevant is defined ostensively in the form of either test items (type III) or a description of a sub-class of educational goals (type II). The reader of Handbook I must then abstract what is relevant to the meaning of the terms by noting what is common to two or more test items and, at a higher level of generality, what two or more sub-classes have in common when they collectively define a class.

The function of operational definitions is not to add empirical significance, but rather to restrict and preserve meaning, as empirical significance, which is presented in each instance ostensively defined. The focus is on student behavior in making a response or manifesting an ability rather than on that which the response or ability is about. When each instance functions ostensively, the situation is stated hypothetically: if student X performs correctly, then he has functioned at the cognitive level in question. Thus, operationally defined terms are meaningless in and of themselves. Meaning is abstracted from situations which are presented and defined ostensively. But, since the meaning of terms in the taxonomy includes both the problematic situation and correct responses to the stimulus situation, operational definitions are necessary for the retention and formulation of empirical significance in the definiens of verbal or type I definitions.

In order to discuss how the classification scheme is explicated, a functional distinction was made between meaning and reference. Connotation, significance, and description were treated as subtopics within meaning. And denotation, naming, and truth were subsumed under reference. The following questions are representative of those discussed: Do ostensive definitions present one instance of a term's denotation? Is the hierarchy of class and sub-class terms true because it is generated from empirical data? and is it empirically verifiable? Do terms in the taxonomy function as logically proper names which have no connotation? If the classification scheme is meaningful, how is meaning derived? What are the conditions for stating verbal (definiendum-definiens) definitions?

It is concluded that since the data to be classified are teacher intended student behaviors, given instances do not necessarily represent what the terms denote. The possibility cannot be overlooked that the terms may have zero denotation. Hence, the claim cannot be made that the taxonomy is all descriptive, i.e., that there are no prescriptive intentions. On the other hand, if meaning, as empirical significance, is abstracted from instances which are descriptive of empirical events, reduced, and formulated as definiens, which are interchangeable with the terms to be defined, then the class and sub-class words are not arbitrarily chosen names, but rather they are terms which connote (meaning) those defining characteristics which are empirically generated (reference).

If the reader of Handbook I reaches the point that verbal definitions are minimally vague and maximally meaningful, for him, then he can deal with the terms involved on a completely symbolic level. Metatheoretically, he is functioning in the area of meaning rather than that of reference. If this is the case, a level of abstraction has been reached at which terms and relations can be considered as constituting a self-contained unit. The abstracted scheme can then be examined in relation to the guiding principles.

There are two types of guiding principles: (1) Those which state requirements that the abstracted taxonomy should meet to a greater or lesser extent. And (2) those of an operational nature which specify conditions that are either met or not met. The first type are such that an assessment could be made of the extent to which the taxonomy meets the stated conditions; i.e., limiting conditions. The second type, on the other hand, specify conditions, instances of which are either met or not met, but the number of instances is not fixed. Specifically, the requirement of descriptiveness could probably be completely met. Prescriptive educational goals could be excluded by defining how the sample is to be taken. Also the taxonomy may be so organized that divisions in the abstracted scheme reflect divisions in the empirical data and do not contain divisions which are not manifested in the data to be classified. The requirements of mutual exclusion, exhaustive sub-classes, internal consistency and completeness are logical criteria to be imposed on the completely abstracted hierarchy. The extent to which these conditions may be met is a function of the extent to which the abstracted taxonomy is self-contained. In the case of external consistency, it would be impossible to state the extent to which the criterion is met, since research in other areas is an on-going matter. In the specific area of psychological research, as theories emerge and become structured, inconsistencies with the taxnomy may be encountered. But this is not a matter of degrees. Either research findings are consistent with the terms and relations defined in Handbook I or they are not. The difference between internal and external consistency is that the former would be inconsistencies with respect to defining criteria, while the latter would be inconsistencies with respect to facts determined by empirical research.

Summary: Philosophical Aspects

It is not always easy to identify the philosophical aspects which might emerge from a problem of the type to which this dissertation has been directed. For the purposes of this writing, inquiry at the level of theory is controlled by those meta-theoretic areas which have been abstracted, organized, and structured as self-contained fields. These areas of meta-theory contain many philosophical problems. A discussion could be presented of the various ways which different authors deal with form and function, meaning and reference, etc. For example, this writing has been more influenced by the works of Quine and Lewis as opposed to Carnap and others who take a more formalistic view. However, a discussion of this kind would be of more interest to logicians than to philosophers of education. Therefore, the philosophical aspects here considered will be those raised in chapter I. That is, how can the taxonomy as an abstracted, empirically significant, theoretical work, be compared to the conception of education as a self-corrective, reconstructive process.

First, a distinction must be made between 'philosophy' as speculative, and 'philosophy' in the tradition of Logical Empiricism. The ground for philosophy is experience. It is a tautology to say that all of experience must be within experience. But are we equating experience with what has been experienced in the world of objects and events? Or does the word 'experience' also apply to hypothetical states of affairs which may never be experienced in actuality? No attempt will be made to answer this question here. It is sufficient merely to note that in some way the realm of what is consistently possible or thinkable, is based upon the world of actual events which have occurred. This is the same distinction as that between connotation and denotation. Meaning is based upon reference in the sense that what we can define does not have to exist but certainly what exists should be definable (describable). Therefore the central problem is whether, or in what way, the existence of referents places restrictions on meaning. In the taxonomy, restrictions were in the form of operational definitions. But we do not have these controls placed on inquiry in general. That is, what we call speculative philosophy may consist in inquiry which is limited only by what is in some way meaningful. There may be few or no empirical restraints.

The question might be better put into focus by making a distinction between means and ends or between process and product. Educational objectives as substantive, utopian goals are presented as ends to be attained -- as the product of the educational process. On the other hand, the goal of education as the constant reorganization and reconstruction of experience is within the conception of education as a process rather than a product.

Is the taxnnomy defined in terms of process or product? Means or ends? The emphasis throughtout this dissertation has been on the necessity of retaining the significance in type I definitions which originated in what was ostensively defined. Thus meaning is a function of intended student behaviors. And the emphasis in type II and type III is on subject matter: on performance with respect to problems and to kinds of problems. Thus, educational goals in the cognitive domain are defined in terms of *that which cognitive processes are about*: i.e., in terms of products.

It might be argued that thinking cannot be defined without reference to acts of thought. And certainly no claim is being made that the taxonomy should be defined without reference to empirical data. But this merely shows that the difference between process and product is one of degree rather than of kind. To claim that cognitive acts could be meaningfully defined and are conceptually significant apart from any overt behavior is unscientific at least and absurd at most. Nevertheless, when educational goals are defined in terms of products, there is the possibility that the dynamic, reconstructive quality which characterizes the learner as more self-acting and less reacting, will be ignored and even lost.

Perhaps the clearest way to make the process-product distinction with respect to the taxonomy is to inquire how the goal of education as the constant reconstruction of experience can be classified in the scheme. When we say that education is its own end, or that the only goal of education is more education, we have stated an intended goal. But is it in terms of student behavior? Perhaps not. It might be a disguised prescriptive statement: i.e., the goal of education should be, etc. However, the difficulty can be easily circumvented by translating the intention into student behavior. The goal for students is not the simple *prima facie* acceptance of what consitiutes the content of a learning experience. It is rather the disposition to reorganize and reconstruct, to analyze, synthesize, and evaluate, and then to reexamine the problem, to consider alternatives and consequences of alternatives and to synthesize new, more primitive, hypotheses which will be superordinate to previous ones. This, then, is a disposition which would be appropriate at any level in the taxonomy. The question is, however, is this a cognitive disposition?[4] Here the argument comes close to being circular; i.e., the disposition is not cognitive because it is not included in the cognitive domain. But would it be possible to include instances of the kind of disposition in question as teacher intended descriptions and therefore to have them count as cognitive goals to be classified? What we are asking for is an example of thinking about 'thinking'. All examples given in Handbook I are cases of thinking about some subject area, but there is no reason to exclude thinking itself as one area. As was seen earlier, this seems to involve equating the whole to a part. The entire cognitive domain would become the object of a learning experience rather than the universe of discourse in terms of which all cognitive activity occurs. The problem is therefore seen to be the same as that which was discussed in chapter IV in terms of completion.

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[4] One way to deal with the problem, at this point, is to say that the disposition is only another name for motivation, which is a pre-condition for cognition but which is itself not in the cognitive domain.

Thus, if the taxonomy does not include itself as a part, it is not complete. To define educational goals in terms of ends or products is to restrict the range of our educational sights. But when we focus on the process itself, as a goal (the end of education is more education), we have set our sights at maximum range, and we can do no more.

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APPENDICES

APPENDIX A

Organization of Handbook I, Cognitive Domain[1]

Part I

Introduction and Explanation

Foreward
- History
- Problems
- Organizational principles
- Three domains-cognitive, affective, and psychomotor
- Development of the cognitive domain

Chapter 1: The Nature and Development of the Taxonomy
- The taxonomy as a classification device
- What is to be classified
- Guiding principles
- Developing the taxonomy
- The problem of a hierarchy-classification versus taxonomy
- Is the taxonomy a useful tool?

Chapter 2: Educational Objectives and Curriculum Development
- Knowledge as a taxonomy category
- What is knowable
- Justification for the development of knowledge
- Curricular decisions to be made about knowledge objectives
- The nature of abilities and skills
- Arts or skills ÷ knowledge = abilities
- Justification for the development of intellectual abilities and skills

Chapter 3: The Problems of Classifying Educational Objectives and Test Exercises
- Three levels of definition in each taxonomy category
- The classification of objectives
- Test yourself on the classification of objectives
- The problems of classifying test exercises
- Test yourself on the classification of test exercises
- Key to the classification of educational objectives

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[1] Bloom, Benjamin S., and others, *Taxonomy of Educational Objectives Handbook I: Cognitive Domain*, David McKay & Co., New York, Quoted with permission of the publisher.

APPENDIX B

Condensed Version of the Taxonomy of Educational Objectives[1]

Cognitive Domain

KNOWLEDGE

1.00 KNOWLEDGE

Knowledge, as defined here, involves the recall of specifics and universals, the recall of methods and processes, or the recall of a pattern, structure, or setting. For measurement purposes, the recall situation involves little more than bringing to mind the appropriate material. Although some alteration of the material may be required, this is a relatively minor part of the task. The knowledge objectives emphasize most the psychological processes of remembering. The process of relating is also involved in that a knowledge test situation requires the organization and reorganization of a problem such that it will furnish the appropriate signals and cues for the information and knowledge the individual possesses. To use an analogy, if one thinks of the mind as a file, the problem in a knowledge test situation is that of finding in the problem or task the appropriate signals, cues, and clues which will most effectively bring out whatever knowledge is filed or stored.

1.10 *KNOWLEDGE OF SPECIFICS*

The recall of specific and isolable bits of information. The emphasis is on symbols with concrete referents. This material, which is at a very low level of abstraction, may be thought of as the elements from which more complex and abstract forms of knowledge are built.

1.11 KNOWLEDGE OF TERMINOLOGY

Knowledge of the referents for specific symbols (verbal and non-verbal). This may include knowledge of the most generally accepted symbol referent, knowledge of the variety of symbols which may be used for a single referent, or knowledge of the referent most appropriate to a given use of a symbol.

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[1] Bloom, Benjamin, et al., Taxonomy of Educational Objectives, Handbook I: Cognitive Domain, David McKay and Co., New York, 1956, pp. 201-207. Quoted by permission of the publisher.

*To define technical terms by giving their attributes, properties, or relations.

*Familiarity with a large number of words in their common range of meanings.

1.12 KNOWLEDGE OF SPECIFIC FACTS

Knowledge of dates, events, persons, places, etc. This may include very precise and specific information such as the specific date or exact magnitude of a phenomenon. It may also include approximate or relative information such as an approximate time period or the general order of magnitude of a phenomenon.

* The recall of major facts about particular cultures.

* The possession of a minimum knowledge about the organisms studied in the laboratory.

1:20 KNOWLEDGE OF WAYS AND MEANS OF DEALING WITH SPECIFICS

Knowledge of the ways of organizing, studying, judging, and criticising. This includes the methods of inquiry, the chronological sequences, and the standards of judgment within a field as well as the patterns of organization through which the areas of the fields themselves are determined and internally organized. This knowledge is at an intermediate level of abstraction between specific knowledge on the one hand and knowledge of universals on the other. It does not so much demand the activity of the student in using the materials as it does a more passive awareness of their nature.

1:21 KNOWLEDGE OF CONVENTIONS

Knowledge of characteristic ways of treating and presenting ideas and phenomena. For purposes of communication and consistency, workers in a field employ usages, styles, practices, and forms which best suit their purposes and/or which appear to suit best the phenomena with which they deal. It should be recognized that although these forms and conventions are likely to be set up on arbitrary, accidental, or authoritative bases, they are retained because of the general agreement or concurrence of individuals concerned with the subject, phenomena, or problem.

---

*Illustrative educational objectives selected from the literature.

*Familiarity with the forms and conventions of the major types of works, e. g., verse, plays, scientific papers, etc.

*To make pupils conscious of correct form and usage in speech and writing.

1:22 KNOWLEDGE OF TRENDS AND SEQUENCES

Knowledge of the processes, directions, and movements of phenomena with respect to time.

*Understanding of the continuity and development of American culture as exemplified in American life.

*Knowledge of the basic trends underlying the devel opment of public assistance programs.

1:23 KNOWLEDGE OF CLASSIFICATIONS AND CATEGORIES

Knowledge of the classes, sets, divisions, and arrangements which are regarded as fundamental for a given subject field, purpose, argument, or problem.

*To recognize the area encompassed by various kinds of problems or materials.

*Becoming familiar with a range of types of literature.

1:24 KNOWLEDGE OF CRITERIA

Knowledge of the criteria by which facts, principles, opinions, and conduct are tested or judged.

*Familiarity with criteria for judgment appropriate to the type of work and the purpose for which it is read.

*Knowledge of criteria for the evaluation of recreational activities.

1.25 KNOWLEDGE OF METHODOLOGY

Knowledge of the methods of inquiry, techniques, and procedures employed in a particular subject field as well as those employed in investigating particular problems and phenomena. The emphasis here is on the individuals' knowledge of the method rather than his ability to use the method.

*Knowledge of scientific methods for evaluating health concepts.

*The student shall know the methods of attack relevant to the kinds of problems of concern to the social sciences.

1.30 KNOWLEDGE OF THE UNIVERSALS AND ABSTRACTIONS IN A FIELD

Knowledge of the major schemes and patterns by which phenomena and ideas are organized. These are the large structures, theories, and generalizations which dominate a subject field or which are quite generally used in studying phenomena or solving problems. These are at the highest levels of abstraction and complexity.

1.31 KNOWLEDGE OF PRINCIPLES AND GENERALIZATIONS

Knowledge of particular abstractions which summarize observations of phenomena. These are the abstractions which are of value in explaining, describing, predicting, or in determining the most appropriate and relevant action or direction to be taken.

*Knowledge of the important principles by which our experience with biological phenomena is summarized.

*The recall of major generalizations about particular cultures.

1.32 KNOWLEDGE OF THEORIES AND STRUCTURES

Knowledge of the body of principles and generalizations together with their interrelations which present a clear, rounded, and systematic view of a complex phenomenon, problem, or field. These are the most abstract formulations, and they can be used to show the interrelation and organization of a great range of specifics.

*The recall of major theories about particular cultures.

*Knowledge of a relatively complete formulation of the theory of evolution.

INTELLECTUAL ABILITIES AND SKILLS

Abilities and skills refer to organized modes of operation and generalized techniques for dealing with materials and problems. The materials and problems may be of such a nature that little or no specialized and technical information is required. Such information as is required can be assumed to be part of the individual's general fund of knowledge. Other problems may require specialized and technical information at a rather high level such that specific knowledge and skill in dealing with the problem and the materials are required. The abilities and skills objectives emphasize the mental processes of organizing and reorganizing material to achieve a particular purpose. The materials may be given or remembered.

2.00 COMPREHENSION

This represents the lowest level of understanding. It refers to a type of understanding or apprehension such that the individual knows what is being communicated and can make use of the material or idea being communicated without necessarily relating it to other material or seeing its fullest implications.

2.10 TRANSLATION

Comprehension as evidenced by the care and accuracy with which the communication is paraphrased or rendered from one language or form of communication to another. Translation is judged on the basis of faithfulness and accuracy, that is, on the extent to which the material in the original communication is preserved although the form of the communication has been altered.

*The ability to understand non-literal statements (metaphor, symbolism, irony, exaggeration).

*Skill in translating mathematical verbal material into symbolic statements and vice versa.

2.20 INTERPRETATION

The explanation or summarization of a communication. Whereas translation involves an objective part for part rendering of a communication, interpretation involves a reordering, rearrangement, or a new view of the material

*The ability to grasp the thought of the work as a whole at any desired level of generality.

*The ability to interpret various types of social data.

2.30 EXTRAPOLATION

The extension of trends or tendencies beyond the given data to determine implications, consequences, corollaries, effects, etc., which are in accordance with the conditions described in the original communication.

*The ability to deal with the conclusions of a work in terms of the immediate inference made from the explicit statements.

*Skill in predicting continuation of trends.

3.00 APPLICATION

The use of abstractions in particular and concrete situations. The abstractions may be in the form of general ideas, rules of procedures, or generalized methods. The abstractions may also be technical principles, ideas, and theories which must be remembered and applied.

*Application to the phenomena discussed in one paper of the scientific terms or consepts used in other papers.

*The ability to predict the probably effect of a change in a factor on a biological situation previously at equilibrium.

4.00 ANALYSIS

The breakdown of a communication into its constituent elements or parts such that the relative hierarchy of ideas is made clear and/or the relations between the ideas expressed are made explicit. Such analyses are intended to clarify the communication, to indicate how the communication is organized, and the way in which it manages to convey its effects, as well as its basis and arrangement.

4.10 ANALYSIS OF ELEMENTS

Identification of the elements included in a communication.

*The ability to recognize unstated assumptions.

*Skill in distinguishing facts from hypotheses.

4.20 ANALYSES OF RELATIONSHIPS

The connections and interactions between elements and parts of a communication.

*Ability to check the consistency of hypotheses with given information and assumptions.

*Skill in comprehending the interrelationships among the ideas in a passage.

4.30 ANALYSIS OF ORGANIZATIONAL PRINCIPLES

The organization, systematic arrangement, and structure which hold the communication together. This includes the "exhibit" as well as "implicit" structure. It includes the bases, necessary arrangement, and the mechanics which make the sommunication a unit.

*The ability to recognize form and pattern in literary or artistic works as a means of understanding their meaning.

*Ability to recognize the general techniques used in persuasive materials, such as advertising, propaganda, etc.

5.00 SYNTHESIS

The putting together of elements and parts so as to form a whole. This involves the process of working with pieces, parts, elements, etc., and arranging and combining them in such a way as to constitute a pattern or structure not clearly there before.

5.10 PRODUCTION OF A UNIQUE COMMUNICATION

The development of a communication in which the writer or speaker attempts to convey ideas, feelings, and/or experiences to others.

*Skill in writing, using an excellent organization of ideas and statements.

*Ability to tell a personal experience effectively.

5.20 PRODUCTION OF A PLAN, OR PROPOSED SET OF OPERATIONS

The development of a plan of work or the proposal of a plan of operations. The plan should satisfy requirements of the task which may be given to the student or which he may develop for himself.

*Ability to propose ways of testing hypotheses.

*Ability to plan a unit of instruction for a particular teaching situation.

5.30 DERIVATION OF A SET OF ABSTRACT RELATIONS

The development of a set of abstract relations either to classify or explain particular data or phenomena, or the deduction of propositions and relations from a set of basic propositions or symbolic representations.

*Ability to formulate appropriate hypotheses based upon an analysis of factors involved, and to modify such hypotheses in the light of new factors and considerations.

*Ability to make mathematical discoveries and generalizations.

6.00 EVALUATION

Judgments about the value of material and methods for given purposes. Quantitative and qualitative judgments about the extent to which material and methods satisfy criteria. Use of a standard of appraisal. The criteria may be those determined by the student or those which are given to him.

6.10 JUDGMENTS IN TERMS OF INTERNAL EVIDENCE

Evaluation of the accuracy of a communication from such evidence as logical accuracy, consistency, and other internal criteria.

*Judging by internal standards, the ability to assess general probability of accuracy in reporting facts from the care given to exactness of statement, documentation, proof, etc.

* The ability to indicate logical fallacies in arguments.

6.20 JUDGMENTS IN TERMS OF EXTERNAL CRITERIA

Evaluation of material with reference to selected or remembered criteria.

*The comparison of major theories, generalizations, and facts about particular cultures.

*Judging by external standards, the ability to compare a work with the highest known standards in its field--especially with other works of recognized excellence.

APPENDIX II

SCALE FOR APPRAISING A MODEL

APPENDIX 2

Scale for Appraising a Model

A. Descriptions

1. Logical Fertility Probability of leading to new observations and experiments. Can be assessed before the event only in terms of subjective expectations. Historians are better able to assess this after some time has passed.

2. Originality Inverse of the probability of the scheme being found within a preceding framework of knowledge. Analyze scheme into its chief elements, and then these into subsets, until all subsetscould be matched with similar ones in existing literature. Then moving upward, one could ask, which combinations of these familiar components are unfamiliar. Another way would be to estimate probability of matching the scheme from successive samples of relevant literature.

3. Organizing power Number of independently established classes of phenomena included. Counting can be done in many ways. To compare theories on organizing power it would be necessary to standardize in some way the counting method. However, past history indicates that the greater generality of a theory becomes evident despite different ways of counting.

4. Predictive range Number of classes of verifiable predictions; Difficult to define; again, a fairly plausible method of counting classes within the prevailing scientific practice

in a field, and holding it constant among comparisons should be satisfactory.

5. **Structural Complexity** Number of categories or variables used.

6. **Functional Complexity** Number of internal operating rules. Internal rules must be distinguished from operational rules of observation or experimentation through which classificatory decisions and factual predictions are linked to reproducible data.

7. **Performance** Number of new phenomena included.

8. **Predictive novelty** Number of new predictions it can generate.

9. **Structural novelty** Number of new categories suggested.

10. **Functional novelty** Number of new operating rules developed and implied.

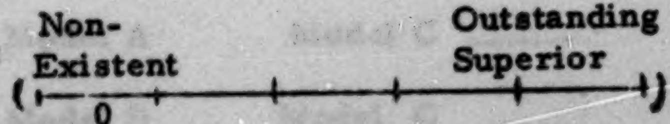
11. **Predictive accuracy** Margin of error, usually a probability distribution. Disparity between observation and prediction code (in psychological amortization).

12. **Self-transcendence** Probability of leading to a new and better scheme.

B. Scale for Appraising a Model

1. **Rating Scale** (Consists of names of each of the 12 items, followed by a 5-point rating scale.)

Example: 1. **Organizing power:**



A. Your estimate for the field in which the individual works

B. Your estimate for similar fields

2. **Directions Rating Scale**

a. Consider the characteristic or item relative to the least model in the field.

b. Consider the characteristic or item relative to the ideal in the field

c. Indicate your impression in the appropriate space on the blank first for the "a" and then the "b" condition.

Example

Item 1. Logical Fertility

None	Below Average	Average for the field	Good	Superior Outstanding
				(No model exceeds it on this.)

2. Originality

None	Below Average	Average for the field	Good	Superior Outstanding

•
•
•

12. Self-transcendence

None	Below Average	Average	Good	Outstanding
			(Better than most)	

3. Paired Comparison Scale (Used to compare two models pair-wise on the 12 characteristics)

Directions: In each instance indicate which model is preferable on the indicated characteristic.

Example:

1. Logical Fertility ____ Model A Model B ____

____ Model A Model C ____

____ Model B Model C ____

etc.

APPENDIX III

EXPLORATION OF INTERACTIONS AMONG INSTRUCTIONAL, CONTENT AND APTITUDE VARIABLES

APPENDIX 3

Exploration of Interactions Among Instruction, Content and Aptitude Variables

A. Correlational procedures:

Correlational methods have been widely used for studying relationships, and while they give much useful information, they do not quite tell the story of interactions. The general use of correlations has been for direct prediction. Often they have given users a false sense of security--they fail to determine why the predictions are as they are. To explore the question of "why" one must go into the details of the interactions, and this means not only exploring strategies in learning and instruction but also exploration of the nature and type of interactions among content, instruction, and aptitude variables. Programmed learning, because all its stimuli are objectively specified, and associated recording equipment (for automatic recording of responses) can begin to provide better data relative to the variables and interactions involved in the instructional process.

This control includes control of content (both in terms of subject matter and in terms of taxonomic variables) and teacher and pupil instrumental responses. For the first time it is now possible to begin talking rigorously about instruction because the details can be objectively spelled out and open to detailed inspection by qualified people. There is no longer need for a large element of surprise that occurs in unique individual interpretations of vague or undefined terms and concepts. Accurate communication and independent verification can thus be performed.

The strategy for detailed study of interactions appears to be analogous, in several respects, to that used in biological research on drugs, vitamins, sera, and other forms of stimuli. Such work begins with biological assay where the stimulus effect is determined by the reaction of living organisms. The stimulus is applied to batches of subjects at several levels and the reaction observed. Potency of agents is determined when probits are plotted against dosage. Range of critical dosage or sensitivity, tests of significance, determination of relative potency of several preparations are routine. Dosage experiments can also be carried out in factorial design in an analysis that involves fitting of probit planes. These steps, however, are but preliminaries to the more complex case--where the relationship of concern is the effect as a joint function of two or more biological preparations.

Maccia and others (14) discuss pharmacology and pharmacogenetics as a possible educational theory model. Much of their discussion revolves about the concept of isolobes, a term introduced by Lowe in 1928. These techniques permit the joint relationships between two drugs in producing an effect to be analyzed and clarified. The number of variables is reduced to two by choosing a specified effect, usually ED 50 (median effective dose).

Varying amounts of the two drugs which jointly produce the specified effect are determined. The plot, or isolobe, of these amounts, on a bivariate chart indicates the combinations required for the identical effect.

In the present study, the taxonomy provides rubrics for content, and calibration in terms of useful units can be performed in terms of probits.

Isolobes have been classified into three types and seven subtypes by De Jongh (7) and summarized by Maccia and other (14).

Type I includes those curves which have intercepts on both axes. If the plot is a straight line, no interaction occurs. One drug can be specified in equivalent amounts freely for the other to produce the specified effect. This subtype is called addition.

If combining requires less of the two drugs than in addition, the plot is called potentiation. When interaction is inhibitory, e.g., it requires more than in addition, the plot is called depotentiation.

Type 2 concerns curves that have only one intercept. Only one drug alone can produce the desired effect. If the amount of drug 1 required is the same whether drug 2 is given or not, the term neutrality is used. When the presence of drug 2 increases the first drug's effect, the term is augmentation. If drug inhibits the first drug's effect, the result is called antagonism. A special case of antagonism occurs when the presence of sufficient drug 2 completely inhibits the specified effect. This is called blockage.

Type III isolobes have no intercept. Neither drug 1 nor drug 2 alone can produce the specified effect. When both are required, the term used is coalitive synergism.

These three types and seven subtypes completely define the domain of two drug interactions.

Interactions among the instructional, content, and aptitude variables can be described similarly in pair-wise fashion. Achievement on the taxonomic classification involved is held constant at the ED 50 level for most comparisons of interaction. Units of the two dimensions of a plot are determined from probit lines or planes, as necessary. Isolobes are plotted from these data. With twenty divisions of content, five of instruction, and five of aptitude, 1,500 isolobes will be plotted. Given the four situations--two units in two subject areas--this requires 6,000 plots.

While this is, of course, a formidable task, it indicates in part the extent of information that we need and are lacking in order to achieve even the crudest sort of specific description of instructional interactions. It adds emphasis to Paul Rosenbloom's (5) comments at the Second Phi Delta Kappa Symposium on Educational Research:

> It is scandalous that basic decisions on educational policy have been made nationally on the most flimsy evidence. For example, the grade placement of topics in arithmetic now universally used in American schools is based on the 1927 study of the Committee of Seven, which was statistically incompetent. We use more care in making decisions about breeding hogs than about teaching children.

After the interactions are plotted, exploration in the development of systematic theories that use the given inputs (of content, aptitude, and instruction variables) to generate as outputs the interactions that were empirically described by the isolobes can proceed. In this connection the tools of simulation can probably play a useful part.

Simulation requires development of a model that describes the operation of the observed phenomena in terms of inputs and outputs. Such models are described by Borko (3), Tomkins and Messick (17), Hunt (13), Coulson (6), Fattu and Robinson (10), Machol (15) has stated that "it is possible to describe analytically any known function which can be reasonably defined in objective terms." He includes teaching among these functions.

IMPLICATIONS

The utility of such models probably will not be immediate. Perhaps the most significant immediate results may be the stimulus to study instruction in depth. The teaching process is exceedingly intricate and complex in its own right, and the complexity is compounded by being hedged in with all sorts of pressures--local and regional, social and political, financial and professional--to name but a few.

For immediate purposes the utility of models of this sort is in research. In the long run they appear to be useful for adding to the fund of knowledge about interactions among instruction, content, and aptitude variables.

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APPENDIX IV

AN ALTERNATIVE TO ISOLOBES FOR EXPLORING TRADE-OFF RELATIONSHIPS AMONG CONTENT, INSTRUCTION, APTITUDE AND ACHIEVEMENT VARIABLES

APPENDIX 4

For this example, achievement score G-7, with H and I irrelevant. (noted by a zero), has been chosen.

Example:

A	B	C	D	E	F	G	H	I
1	3	9	2	1	1	7	0	0
1	3	9	3	2	2	7	0	0
3	4	9	5	1	2	7	0	0
4	4	8	5	2	3	7	0	0
4	4	8	5	3	5	7	0	0
5	5	8	5	2	6	7	0	0
6	5	8	5	2	7	7	0	0
7	5	7	6	3	6	7	0	0
7	6	7	6	2	6	7	0	0
7	8	6	7	2	7	7	0	0
8	8	5	7	4	8	7	0	0
9	9	4	8	3	9	7	0	0
9	9	2	7	1	9	7	0	0

Columns are vertical, rows are horizontal, and A: (1-5) will mean column A, items 1 thru 5, i.e., 1, 1, 3, 4, 4.

1. Sort on A. In the example above, this step has already been performed. That is, A is in ascending order. For these purposes, the segment consisting of the items 3, 4, 4, 5, is considered to be in ascending order; i.e., a value may be duplicated within an ascending order.
2. Check columns for directionality (that is, are the items on the list in an ascending or descending order). Label directional sections of the columns if and only if they consist of 4 or more contiguous items arranged in ascending or descending order. All other

sections, sections which do not consist of 4 or more contiguous items arranged in ascending or descending order, will be labelled FUNNY. In the example,

B: ASC (1 - 12)
C: DSC (1 -13)
D: ASC (1 - 12), D: FUNNY (13)
E: FUNNY (1 -13)
F: ASC (1 - 7), F: ASC (8 -13)

Column E is labelled FUNNY for all of its items; i.e., there is no section which consists of at least 4 contiguous items which displays ascending or descending order. Any FUNNY section within (or co-terminous with) a column opens the possibility that that column in which it occurs should be sorted later; in fact, only FUNNY sections whose items constitute over half of the items in a given column will cause their columns to be sorted later. In the example, column E's FUNNY section will cause that column to be sorted later; and column D's FUNNY section will not, as it does not contain over half of the items in that column.

3. Note all uni-directional sections (consisting of at least 4 items) both individually and in combination with other sections from other columns. That is, these are the sections (which may comprise an entire column) which are available for trade-offs. Individually, in the example, we have

I. A: ASC (1 -13)
II. D: ASC (1 - 12)
III. F: ASC (1 - 7)

IV. F: ASC (8 - 13)

V. C: DSC (1 - 13)

Two or more sections which could not be traded off, one against the other (because they are all ascending or all descending), may be combined to create a compound section which is then eligible to enter trade offs on precisely the same basis as any of the individual sections above. There can exist trades between two compound sections, between two individual sections, or between an individual and a compound section. Item by item addition of the values in two (or more) sections will result in the values of the compound section. For example, in column A above, the value of the first item--1--is added to the value of the first item in column D--2--to provide the value of the first item in a compound section. In the example, the values of VI (below) would be:

3, 4, 8, 9, 9, 10, 11, 13, 13, 14, 15, 17

The combined sections are

VI. A: ASC (1 -12) and D: ASC (1 -12)

VII. A: ASC (1 - 7) and F: ASC (1-7)

VIII. A: ASC (1 - 7) and D: ASC (1 - 7) and F: ASC (1 - 7)

IX. A: ASC (8 - 13) and F: ASC (8-13)

X. A: ASC (8 - 12) and D: ASC (8 - 12) and F: ASC (8 - 12)

XI. D: ASC (1 - 7) and F: ASC (1 - 7)

XII. D: ASC (8 - 12) and F: ASC (8 -12)

All of these may be traded off with the relevant section of XIII.

C: DSC (1 -13)

That is, we may have trades between I and XIII, II and XIII, III and XIII, IV and XIII ... XII and XIII. In, for example, the trade between VIII and XIII, only that section of XIII which consists of the first thru the seventh item will be relevant.

Then, for all of the trade possibility pairs, hypothetical step-size will be determined. This will be by the formula, largest value number minus smallest value number all divided by one less than the number of items:

$$\frac{L - S}{n - 1}$$

Then, print out will be obtained on the trade possibilities. Then, the procedure iterates for a sort on all columns which have FUNNY sections which consist of over half of the items in those columns. In the example, a sort would take place at this point on column E, and the entire procedure would be repeated, except that FUNNY sections found on this iteration--even if they constitute over half of the items on their columns--would <u>not</u> cause further iteration.

SAMPLE PRINTOUT:

```
VARIABLE              RANGE                        STEPSIZE

  A, D       A, ASC (1 - 9)  D, DSC (7 -2)  A (0.6600)  D(-0.4166)
```

The program which readies the data for analysis is given below, followed by the analysis program. When this latter program was run on the sample data (given on the first page of this appendix), it performed properly.

```
      PROGRAM CODE1    J. Liston
C*********************************************************************
C
C         COMPUTE CODE/CLASS FROM VARIABLE INPUT
C         METHOD
C         WIDTH = (HIGH - LOW) / NUMBER OF CLASSES OR CODES
C         A BLANK FIELD IS SET TO CLASS OR CODE ZERO
C         THE LOW SCORE IS ASSIGNED TO CLASS OR CODE ONE
C         THE REMAINING CLASSES OR CODES ARE COMPUTED AS FOLLOWS
C         CLASS OR CODE NUMBER = (SCORE - LOW) / WIDTH
C         AND ROUND QUOTIENT TO NEXT HIGHEST INTEGER
C
C*********************************************************************
      DIMENSION  ALOW(100),DIV(100),HIGH(100),QUOT(100),SCORE(100)
      DIMENSION  WIDTH(100),BLANKT(100).CDBCD(10)
      COMMON/BLOCK1/ALOW,DIV,HIGH,QUOT,SCORE,WIDTH,BLANKT,CDBCD
      DATA (ALOW = 100(99999.)), (HIGH = 100(0.))
      ITOTAL = 21
C
C         SET HIGH/LOW SCORE AND CHECK FOR BLANK FIELDS
C         IF A FIELD IS BLANK, SET ITS  VALUE TO A MINUS ONE
C
  100 READ 1010,(CDBCD(I), I = 1,10,1 )
 1010 FORMAT (10A8)
      IF (EOF,60) 300,200
  200 DECODE (80,1020,CDBCD) AID,SEC,CD,(SCORE(I), I = 1,ITOTAL,1)
 1020 FORMAT (F5.F2.F1,4F3.F2,16F3)
      ASSIGN 220 TO ISW1
      DO 5 I2 = 1,ITOTAL,1
      IF(SCORE(I2))210,210,250
  210 GO TO ISW1,(220,230)
  220 DECODE (80,1030,CDBCD) (BLANKT(I), I = 1,ITOTAL,1)
 1030 FORMAT (8X,4R3,R2,16R3)
      ASSIGN 230 TO ISW1
  230 ITEST1 = BLANKT(I2) .AND. 1R
      IF(ITEST1) 240,250,240
  240 SCORE(I2) = -1.
      GO TO 5
  250 IF (SCORE(I2) - HIGH(I2)) 3,3,2
    2 HIGH(I2) = SCORE(I2)
    3 IF (SCORE(I2) - ALOW(I2))4,5,5
    4 ALOW(I2) = SCORE(I2)
    5 CONTINUE
      WRITE (10) AID,SEC,CD,(SCORE(I8),I8 = 1,ITOTAL,1)
      GO TO 100
  300 END FILE 10
      REWIND 10
C
C        SET THE NUMBER OF CODES/CLASSES AND COMPUTE CODE/CLASS WIDTH
C
      DO 7  I4 = 1,ITOTAL,1
      DIV(I4) = 9.
      WIDTH(I4) = (HIGH(I4) - ALOW(I4)) / DIV(I4)
      PRINT 1050,(HIGH(I),ALOW(I),WIDTH(I), I = 1,ITOTAL,1)
 1050 FORMAT (1H ,16X,F9.4,12X,F7.4,13X,F7.4)
C
C        COMPUTE CODE/CLASS ASSIGNMENT
C
      PRINT 1060
 1060 FORMAT (1H1,*INPUT FOR CODING/CLASS ASSIGNMENT*/7/)
  400 READ (10)  AID,SEC,CD,(SCORE(I8),I8 = 1,ITOTAL,1)
      IF(EOF,10) 600,500
  500 PRINT 1070,              AID,SEC,CD,(SCORE(I), I = 1,ITOTAL,1)
 1070 FORMAT (1H ,F5,F2,F1,(25(1X,F3)))
      DO 10 I5 =1,ITOTAL,1
      IF(SCORE(I5)) 40,41,41
   40 QUOT(I5) = 0.
      GO TO 10
   41 IF(SCORE(I5) - ALOW(I5)) 42,42,43
   42 QUOT(I5) = 1.
      GO TO 10
   43 QUOT(I5) = (SCORE(I5) - ALOW(I5)) / WIDTH(I5)
C          CHECK FOR REMAINDER
C          IF REMAINDER, INCREMENT QUOTIENT BY ONE
      IQUOT = QUOT(I5)
      IF(QUOT(I5) - IQUOT) 46,46,45
   45 QUOT(I5) = IQUOT + 1
      GO TO 10
   46 QUOT(I5) = IQUOT
   10 CONTINUE
      WRITE (13,1004) AID,SEC,CD,(QUOT(I9),I9 = 1,ITOTAL,1)
 1004 FORMAT (F5,F2,F1,1X,70F1)
      GO TO 400
  600 END FILE 13
      REWIND 13
C
C        CODED RESULT - INPUT TO PART TWO
C
      PRINT 1090
 1090 FORMAT (1H1,*CODED/CLASS RESULT - INPUT TO PART TWO OF PROGRAM*//)
  700 READ (13,1004)AID,SEC,CD,(QUOT(I15),I15 = 1,ITOTAL,1)
      IF (EOF,13) 800,750
  750 PRINT 1001,AID,SEC,CD,(QUOT(I15),I15 = 1,ITOTAL,1)
 1001 FORMAT (F5,F2,F1,1X,(25(1X,F3)))
      GO TO 700
  800 CALL EXIT
      END
      PROGRAM FATTU1      S. Kent Brown

      COMMON INFORM(500,25),LIST(22,251),IFUNNY(22)
      COMMON ADD1COL,ADD2COL
      DIMENSION FUNNY(23),SEARCH(3),INTERMFU(25),COLORDER(22),
     X   NOWCOL(25),COLUMN(25),SEARCHPM(6)
      DIMENSION COLUMNS1(22),COLUMNS2(22),LOOP1COL(22),LOOP2COL(22),
     X   OUT1COL(22),OUT2COL(22),ADD1COL(500),ADD2COL(500),
     X   ILOW1(22),IHIGH1(22),ILOW2(22),IHIGH2(22),SORT(22)
      INTEGER FUNNY,SEARCH,COLORDER,TCOL,COLUMN,SEARCHPM,ERROR
      INTEGER GOT1,GOT2,COLUMNS1,COLUMNS2,OUT1COL,OUT2COL,ADD1COL
      INTEGER ADD2COL,HILIMIT1,HILIMIT2,GOTNEED1,GOTNEED2
      INTEGER GOTFUNNY,ALLSORT,SORT
      REAL MINUTES
      EQUIVALENCE (COLORDER,IFUNNY),(COLUMN(1),SEARCH(1))
      EQUIVALENCE (COLUMN(4),COLORDER(1))


COMMENT   INPUT TEST SCORE DATA IS CONTAINED ON LOGICAL TAPE 13 IN THE
C         FOLLOWING FORMAT...
C
C             COLUMN  1 -  5      ID FIELD 1
C             COLUMN 10 - 34      TEST SCORE RESULTS (25 NUMBERS)
C             COLUMN 48 - 55      ID FIELD 2
C
C         EITHER AN END OF FILE OR EXCEEDING THE MAXIMUM NUMBER OF
C         PERMISSABLE INPUT TEST SCORE CARDS (500) WILL TERMINATE INPUT
C         FROM LOGICAL TAPE 13.
COMMENT   SEARCH KEY(S) ARE CONTAINED ON THE STANDARD INPUT TAPE
C         (LOGICAL 60).  END OF FILE ON THIS TAPE TERMINATES THE PROGRAM.


      STARTIME = TIMEF(X)
      DO 1000 K = 1,25
1000  NOWCOL(K) = K
      DO 1010 K = 1,500
      READ (13,100) (INFORM(K,N), N = 1,25)
100   FORMAT(9X,25I1)
      IF(EOF,13)  1020,1010
1010  CONTINUE
      ICARDS = 500
      GOTO 1030
1020  ICARDS = K-1
1030  REWIND 13

1040  READ 101, NSEARCH,NCOL,SEARCH,COLORDER,SEARCHPM,ALLSORT
101   FORMAT(34I2)
      GOTFUNNY = 0
      IF(EOF,60)  1050,1060
1050  TTIME = (TIMEF(X)-STARTIME)/1000.
      MINUTES = K = TTIME/60.
      SECONDS = TIME-60.*MINUTES
      PRINT 102, MINUTES,SECONDS
102   FORMAT(*1*//10X,*END OF INPUT*/10X,*RUN STATISTICS*/
     X   12X,F3.0,* MINUTES*/12X,F7.3,* SECONDS*)
      CALL EXIT

1060  ERROR = 0
      PRINT 103, (SEARCH(K), COLORDER(K), K = 1,3),(COLORDER(K),
     X   K = 4,22),(SEARCH(K),SEARCHPM(2*K-1),SEARCHPM(2*K), K = 1,8)
103   FORMAT(*1*///50X,*SEARCH AND ORDERING PARAMETERS*//52X,*SEARCH*,
     X   4X,*ORDERING*//3(54X,I2,9X,I2/),19(65X,I2/)//3(50X,*COLUMN*,I3,
     X   * SEARCH VALUES ACCEPTED FROM*,I3,* TO*,I3,*.*/))
      PRINT 58, (K,NOWCOL(K), K = 1,25)
58    FORMAT(//10X,*ARRAY NOWCOL*/(10X,I2,*.*,I5))
      DO 1090 K = 1,3
      IF(K .GT. NSEARCH)  GOTO 1060
      IF(SEARCH(K) .GT. 0 .AND. SEARCH(K) .LE. 25)  GOTO 1090
1070  ERROR = 1
      PRINT 104
104   FORMAT(/2X,5H******ERROR IN SEARCH PARAMETERS*)
      GOTO 1090
1080  IF(SEARCH(K) .LT. 0 .OR. SEARCH(K) .GT. 25)  GOTO 1070
1090  CONTINUE
1100  DO 1130 K = 1,22
      IF(K .GT. NCOL)  GOTO  1120
      IF(COLORDER(K) .GT. 0 .AND. COLORDER(K) .LE. 25)  GOTO 1130
1110  ERROR = 1
      PRINT 105
105   FORMAT(2X,5H******ERROR IN COLUMN ORDERING PARAMETERS*)
      GOTO 1130
1120  IF(COLORDER(K) .LT. 0 .OR. COLORDER(K) .GT. 25)  GOTO 1110
1130  CONTINUE
1140  IF(NSEARCH .LE. 3 .AND. NSEARCH .GE. 0)  GOTO 1150
      PRINT 106, NSEARCH
106   FORMAT(/2X,5H******INCORRECT NUMBER OF SEARCH PARAMETERS (*,I2,
     X   *).*)
      ERROR = 1
1150  IF(NCOL .LE. 22 .AND. NCOL .GT. 0)  GOTO 1160
      PRINT 107, NCOL
107   FORMAT(/2X,5H******INCORRECT NUMBER OF DATA COLUMNS (*,I2,*).*)
      ERROR = 1
1160  DO 1170 K = 1,6
      IF(K .GT. 2*NSEARCH)  GOTO 1170
      IF(SEARCHPM(K) .GE. 1 .AND. SEARCHPM(K) .LE. 9)  GOTO 1170
      PRINT 108, K,SEARCHPM(K)
108   FORMAT(/2X,5H******SEARCH VALUE NUMBER*,I2,* (*,I2,*) IS INCORR*
     X   *ECT.*)
      ERROR = 1
1170  CONTINUE
      IF(ERROR .EQ. 1)  GOTO 1230

COMMENT   DETERMINE WHETHER SEARCH AND REORDER DATA IS LOGICALLY CONSISTANT

      DO 1180 K = 1,25
1180  INTERMED(K) = 0
```

```
      DU 1220 K = 1.25
      IF(COLUMN(K) .EJ. 0)  GOTO 1220
      M = K+1
      DU 1210 N = M,25
      IF(COLUMN(K) .NE. COLUMN(N))  GOTO 1210
      IF(K .GT. 3)  GUTJ 1200
      IF(N .GT. 3)  GOTO 1190
      PRINT 109, COLUMN(K)
109   FURMAT(/2X,5H******COLUMN NUMBER*,I3,* IS USED AS A SEARCH KEY*
     A     * TWICE. THIS IS NOT PERMITTED.*)
      ERROR = 1
      GUTO 1210
1190  PRINT 110, COLUMN(K)
110   FORMAT(/2X,5H******COLUMN NUMBER*,I3,* IS BEING USED AS BOTH A *
     A     *SEARCH KEY AND AS A DATA FILE. THIS IS NOT PERMITTED.*)
      ERROR = 1
      GUTU 1210
1200  PRINT 111, COLUMN(K)
111   FORMAT(/2X,5H******COLUMN NUMBER*,I3,* IS USED AS A DATA FILE *
     A     *TWICF. THIS IS NOT PERMITTED.*)
      ERROR = 1
1210  CONTINUE
      INTERMED(COLUMN(K)) = 1
1220  CONTINUE


      IF(ERROR)  1230,1240
1230  PRINT 112
112   FURMAT(///2X,5H******DESTRUCTIVE ERRORS ARE LISTED ABOVE.*,
     X     * THIS SEARCH IS BYPASSED.*)
      GOTO 1040

COMMENT   REARRANGE COLUMNS ACCORDING TO PARAMETERS GIVEN

1240  IKENT = ICARDS+5
      DO 1270 K = 1,25
      IF(INTERMED(K)) 1270,1250
1250  DO 1260 N = 1,25
      IF(COLUMN(N) .NE. 0)  GOTO 1260
      COLUMN(N) = K
      GUTO 1270
1260  CONTINUE
1270  CONTINUE
      DO 1300 K = 1,ICARDS
      DO 1280 N = 1,25
1280  INTERMED(NOWCOL(N)) = INFORM(K,N)
      DO 1290 N = 1,25
1290  INFORM(K,N) = INTERMED(COLUMN(N))
1300  CONTINUE
      DO 1310 K = 1,25
1310  NOWCOL(K) = COLUMN(K)

COMMENT   DETERMINE ROWS WHICH FIT SEARCH PARAMETERS AND REARRANGE
C         TO MAKE THESE APPEAR FIRST IN ARRAY INFORM
```

```
      ICARD = 0
      DO 1340 K = 1,ICARDS
      DO 1320 N = 1,NSEARCH
      IF(INFORM(K,N) .LT. SEARCHPM(2*N-1) .OR. INFORM(K,N) .GT.
     X   SEARCHPM(2*N))  GOTO 1340
1320  CONTINUE
      ICARD = ICARD+1
      IF(ICARD .EQ. K)  GOTO 1340
      DO 1330 N = 1,25
      INTERMED(1) = INFORM(K,N)
      INFORM(K,N) = INFORM(ICARD,N)
1330  INFORM(ICARD,N) = INTERMED(1)
1340  CONTINUE

COMMENT   SORT ACCEPTED DATA ACCORDING TO COLUMN ISORT

      DO 1345 K = 1,NCOL
1345  SORT(K) = NCOL-K+4
1350  DO 1375 NSORT = 1,NCOL
      ISORT = SORT(NSORT)
      IF(ALLSORT .EQ. 0)  ISORT = SORT(NCOL)
      DO 1370 K = 1,ICARD
      M = K+1
      DO 1370 N = M,ICARD
      IF(INFORM(K,ISORT) .LE. INFORM(N,ISORT))  GOTO 1370
      DO 1360 J = 1,25
      INTERMED(1) = INFORM(K,J)
      INFORM(K,J) = INFORM(N,J)
1360  INFORM(N,J) = INTERMED(1)
1370  CONTINUE
      IF(ALLSORT)  1375,1376
1375  CONTINUE

COMMENT   SEARCH THROUGH INPUT COLUMN VECTORS FOR UNIDIRECTIONAL
C         SEGMENTS AND NOTE ALL FUNNY SECTIONS (LESS THAN 4 CONTIGUOUS
C         ITEMS).  LIST CONTAINS THE INITIAL POSITION AND 2*COUNT OF
C         THESE UNIDIRECTIONAL SEGMENTS (COUNT IS + IF ASCENDING, - IF
C         DESCENDING).  IFUNNY CONTAINS A RUNNING TOTAL OF FUNNY ITEMS
C         FOR EACH COLUMN.

1376  NCOL = NCOL+3
      DO 1450 K = 4,NCOL
      J = K-3
      IFUNNY(J) = 0
      IWAY = 0
      LAST = 1
      IZERO = 0
      M = 2
      IPOINTER = 2
      IF(INFORM(1,K) .NE. 0)  GOTO 1377
      LAST = 2
      M = 3
1377  DO 1432 N = M,ICARD
```

```
          IF(INFORM(N,K) .NE. 0)  GOTO 1370
          IZERO = 1
          IF(IWAY)  1420,1400,1400
    1370  IF(IWAY)  1415,1380,1395
    1380  IF(INFORM(N,K)-INFORM(N+1,K)) 1390,1430,1385
    1385  IWAY = 1
          GOTO 1430
    1390  IWAY = -1
          GOTO 1430
    1395  IF(INFORM(N,K) - INFORM(N+1,K)) 1400,1430,1430
    1400  MANY = N-LAST
          IF(MANY .GE. 4) GOTO 1405
          IFUNNY(J) = IFUNNY(J) + MANY
          GOTO 1410
    1405  LIST(J,IPOINTER) = LAST
          LIST(J,IPOINTER+1) = MANY
          IPOINTER = IPOINTER + 2
    1410  LAST = N
          IWAY = 0
          GOTO 1430
    1415  IF(INFORM(N,K) - INFORM(N-1,K)) 1430,1430,1420
    1420  MANY = N-LAST
          IF (MANY .GE. 4) GOTO 1425
          IFUNNY(J) = IFUNNY (J) + MANY
          GOTO 1410
    1425  LIST(J,IPOINTER) = LAST
          LIST (J,IPOINTER+1) = -MANY
          IPOINTER = IPOINTER + 2
          GOTO 1410
    1430  IF(IZERO .EQ. 0)  GOTO 1432
          N = N+1
          LAST = LAST+1
          IZERO = 0
    1432  CONTINUE
          LIST(J,1) = IPOINTER - 1
          MANY = ICARD+1 - LAST
          IF(MANY .LT. 4) GOTO 1445
          LIST(J,IPOINTER) = LAST
          LIST(J,1) = LIST(J,1)+2
          IF(IWAY) 1435,1440,1440
    1435  LIST(J,IPOINTER+1) = -MANY
          GOTO 1450
    1440  LIST(J,IPOINTER+1) = MANY
          GOTO 1450
    1445  IFUNNY(J) = IFUNNY(J) + MANY
    1450  CONTINUE
          NCOL = NCOL-3

    COMMENT   DETERMINE WHICH COLUMNS WILL COMPRISE THE NEXT SORT KEY(S),
    C         IE. FUNNY SECTIONS COMPRISE MORE THAN HALF OF AN ENTIRE COLUMN.

          IF(GOTFUNNY .NE. 0)  GOTO 1460
          J = 2
          N = ICARD/2+1
          DO 1460 K = 1,NCOL
```

```
      IF(IFUNNY(K) .LT. N)  GOTO 1460
      FUNNY(J) = K
      J = J+1
1460  CONTINUE
      FUNNY(1) = J-2


COMMENT   BEGIN SEARCH FOR TRADE-OFF POSSAVILITIES

      PRINT 113
113   FORMAT(*1*,2(/),25X,*FOR THE DESIRED SEARCH RANGES  TRADEOFFS*,
     X    * ARE POSSIBLE IN THE FOLLOWING CASES*)
      LINE = 5
1480  DO 1490 K = 1,500
1490  ADD1COL(K) = ADD2COL(K) = 0

      DO 1770 NEED1 = 1,NCOL
      GOTNEED1 = 0
      DO 1750 NEEJ2 = 1,NCOL
      GOTNEED2 = 0

COMMENT   K LOOP

      KCOL = 0
      GOT1 = 1

1500  KCOL = COLUMNS1(GOT1) = KCOL+1
      IF(KCOL+NEED1-GOT1 .GT. NCOL)  GOTO 1720
      IF(LIST(KCOL,1) .LT. 4)  GOTO 1500
      N = LOOP1COL(GOT1) + 3
      OUT1COL(GOT1) = NONCOL(KCOL+3)
      GOTO (1520,1540)  GOT1

1510  N = LOOP1COL(GOT1) = N+2
      IF(N .GT. LIST(KCOL,1))  GOTO 1500
      GOTO (1520,1540)  GOT1

1520  ISIGN1 = LIST(KCOL,N)
      ILOWER1 = 0
      IUPPER1 = 10000

1540  IX = LIST(KCOL,N)
      IF(IX .GT. 0 .AND. ISIGN1 .LT. 0 .OR.
     X   IX .LT. 0 .AND. ISIGN1 .GT. 0)  GOTO 1510
      ISTART = LIST(KCOL,N+1)
      IEND = XABSF(IX)+ISTART-1
      IF(ISTART .GE. ILOWER1)  GOTO 1550
      IF(IEND .LE. ILOWER1)  GOTO 1510
      GOTO 1555
[illegible] IF(ISTART .GE. IUPPER1)  GOTO 1590
```

```
      DO 1560 IQ = ILOWER1,IUPPER1
1560  ADD1COL(IQ) = ADD1COL(IQ)+INFORM(IQ,KCOL+3)
      GOT1 = GOT1+1
      IF(GOT1 .GT. NEED1)  GOTO 1570
      GOTO 1500

COMMENT   J LOOP

1570  JCOL = COLUMNS1(1)
      GOT2 = 1

1580  JCOL = COLUMNS2(GOT2) = JCOL+1
      IF(JCOL+NEED2-GOT2 .GT. NCOL)  GOTO 1680
      IF(LIST(JCOL,1) .LT. 3)  GOTO 1580
      N = LOOP2COL(GOT2) = 3
      OUT2COL(GOT2) = NOWCOL(JCOL+3)
      GOTO (1600,1620)  GOT2

1590  N = LOOP2COL(GOT2) = N+2
      IF(N .GT. LIST(JCOL,1))  GOTO 1580
      GOTO (1600,1620)  GOT2

1600  ISIGN2 = LIST(JCOL,N)
      IF(ISIGN1 .GT. 0 .AND. ISIGN2 .GT. 0 .OR.
     X   ISIGN1 .LT. 0 .AND. ISIGN2 .LT. 0)  GOTO 1590
      ILOWER2 = ILOWER1
      IUPPER2 = IUPPER1

1620  IX = LIST(JCOL,N)
      IF(IX .GT. 0 .AND. ISIGN2 .LT. 0 .OR.
     X   IX .LT. 0 .AND. ISIGN2 .GT. 0)  GOTO 1590
      ISTART = LIST(JCOL,N-1)
      IEND = XABSF(IX)+ISTART-1
      IF(ISTART .GE. ILOWER2)  GOTO 1630
      IF(IEND .LE. ILOWER2)  GOTO 1590
      GOTO 1635
1630  IF(ISTART .GE. IUPPER2)  GOTO 1580
1635  ILOWER2 = ILOW2(GOT2) = XMAX0F(ISTART,ILOWER2)
      IUPPER2 = IHIGH2(GOT2) = XMIN0F(IEND,IUPPER2)
      DO 1640 IQ = ILOWER2,IUPPER2
1640  ADD2COL(IQ) = ADD2COL(IQ)+INFORM(IQ,JCOL+3)
      GOT2 = GOT2+1
      IF(GOT2 .LE. NEED2)  GOTO 1580

COMMENT   WRITE OUT RESULTS

      STEP = IUPPER2-ILOWER2
      STEP1 = (ADD1COL(IUPPER2)-ADD1COL(ILOWER2))/STEP
      STEP2 = (ADD2COL(IUPPER2)-ADD2COL(ILOWER2))/STEP
      PRINT 116, (OUT1COL(N), N = 1,NEED1)
116   FORMAT(/10X,*GROUP 1 COLUMN(S) =*,22(I3,*,*))
      PRINT 117, (OUT2COL(N), N = 1,NEED2)
      X   ADD2COL(IUPPER2),STEP1,STEP2
 118   FORMAT(10X,*GROUP 1 RANGE (*,I4,*--*,I4,*)       GROUP 2 RANGE (*,
      X   I4,*--*,I4,*)       GROUP 1 STEPSIZE (*,F8.3,*)       *
      X   *GROUP 2 STEPSIZE (*,F8.3,*)*)
       GOTNEED1 = GOTNEED2 = 1
       LINE = LINE+4
       IF(LINE .LE. 55)  GOTO 1650
       PRINT 115
 115   FORMAT(*1*)
       LINE = 0

COMMENT   INCREMENT SEARCH PARAMETERS

 1650. GOT2 = GOT2-1
       ISTART = ILOW2(GOT2)
       IEND = IHIGH2(GOT2)
       DO 1660 IQ = ISTART,IEND
 1660  ADD2COL(IQ) = ADD2COL(IQ)+INFORM(IQ,JCOL+3)
       N = LOOP2COL(GOT2)
       GOTO (1590,1670)  GOT2
 1670  ILOWER2 = ILOW2(GOT2-1)
       IUPPER2 = IHIGH2(GOT2-1)
       GOTO 1590

 1680  IF(GOT2 .EQ. 1)  GOTO 1695
       GOT2 = GOT2-1
       JCOL = COLUMNS2(GOT2)
       ILOWER2 = ILOW2(GOT2)
       IUPPER2 = IHIGH2(GOT2)
       DO 1685 IQ = ILOWER2,IUPPER2
 1685  ADD2COL(IQ) = ADD2COL(IQ)+INFORM(IQ,JCOL+3)
       GOTO 1580

 1695  GOT1 = GOT1-1
       ISTART = ILOW1(GOT1)
       IEND = IHIGH1(GOT1)
       DO 1700 IQ = ISTART,IEND
 1700  ADD1COL(IQ) = ADD1COL(IQ)+INFORM(IQ,KCOL+3)
       N = LOOP1COL(GOT1)
       GOTO (1510,1710)  GOT1
 1710  ILOWER1 = ILOW1(GOT1-1)
       IUPPER1 = IHIGH1(GOT1-1)
       GOTO 1510

 1720  IF(GOT1 .EQ. 1)  GOTO 1740
       GOT1 = GOT1-1
       KCOL = COLUMNS1(GOT1)
       ILOWER1 = ILOW1(GOT1)
       IUPPER1 = IHIGH1(GOT1)
       DO 1725 IQ = ILOWER1,IUPPER1
 1725  ADD1COL(IQ) = ADD1COL(IQ)-INFORM(IQ,KCOL+3)
       GOTO 1500

 1740  IF(GOTNEED2 .EQ. 0)  GOTO 1760
 1760  IF(GOTNEED1 .EQ. 0)  GOTO 1780
 1770  CONTINUE

 1780  GOTFUNNY = GOTFUNNY+1
       IF(GOTFUNNY .GT. FUNNY(1))  GOTO 1840
       N = NCOL-FUNNY(GOTFUNNY+1)+2
       ISORT = SORT(N-1)
       DO 1790 K = N,NCOL
 1790  SORT(K-1) = SORT(K)
       SORT(NCOL) = ISORT
       GOTO 1353
       END PROGRAM FATTU1
```

```
SEARCH AND ORDERING PARAMETERS

  SEARCH    ORDERING

     7          1
     0          2
     0          3
                4
                5
                6
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0
               -0

COLUMN  7 SEARCH VALUES ACCEPTED FROM  7 TO  7.
COLUMN  0 SEARCH VALUES ACCEPTED FROM  1 TO -0.
COLUMN  0 SEARCH VALUES ACCEPTED FROM -0 TO -0.
```

```
FOR THE DESIRED SEARCH RANGES, TRADEOFFS ARE POSSIBLE IN THE FOLLOWING CASES
  GROUP 1 COLUMN(S) =  1,
  GROUP 2 COLUMN(S) =  3,
  GROUP 1 RANGE (   1-   9)     GROUP 2 RANGE (   9-   2)     GROUP 1 STEPSIZE (   0.667)     GROUP 2 STEPSIZE (  -0.583)

  GROUP 1 COLUMN(S) =  2,
  GROUP 2 COLUMN(S) =  3,
  GROUP 1 RANGE (   3-   6)     GROUP 2 RANGE (   9-   7)     GROUP 1 STEPSIZE (   0.429)     GROUP 2 STEPSIZE (  -0.286)

  GROUP 1 COLUMN(S) =  2,
  GROUP 2 COLUMN(S) =  3,
  GROUP 1 RANGE (   5-   9)     GROUP 2 RANGE (   7-   2)     GROUP 1 STEPSIZE (   1.000)     GROUP 2 STEPSIZE (  -1.250)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  4,
  GROUP 1 RANGE (   9-   4)     GROUP 2 RANGE (   2-   8)     GROUP 1 STEPSIZE (  -0.455)     GROUP 2 STEPSIZE (   0.545)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  5,
  GROUP 1 RANGE (   8-   7)     GROUP 2 RANGE (   2-   3)     GROUP 1 STEPSIZE (  -0.333)     GROUP 2 STEPSIZE (   0.333)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  6,
  GROUP 1 RANGE (   9-   8)     GROUP 2 RANGE (   1-   7)     GROUP 1 STEPSIZE (  -0.167)     GROUP 2 STEPSIZE (   1.000)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  6,
  GROUP 1 RANGE (   7-   2)     GROUP 2 RANGE (   6-   9)     GROUP 1 STEPSIZE (  -1.000)     GROUP 2 STEPSIZE (   0.600)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  4,  5,
  GROUP 1 RANGE (   8-   7)     GROUP 2 RANGE (   7-   9)     GROUP 1 STEPSIZE (  -0.333)     GROUP 2 STEPSIZE (   0.667)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  4,  6,
  GROUP 1 RANGE (   9-   8)     GROUP 2 RANGE (   3-  12)     GROUP 1 STEPSIZE (  -0.167)     GROUP 2 STEPSIZE (   1.500)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  4,  6,
  GROUP 1 RANGE (   7-   4)     GROUP 2 RANGE (  12-  17)     GROUP 1 STEPSIZE (  -0.750)     GROUP 2 STEPSIZE (   1.250)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  5,  6,
  GROUP 1 RANGE (   8-   8)     GROUP 2 RANGE (   8-   9)     GROUP 1 STEPSIZE (   0.000)     GROUP 2 STEPSI.E (   1.000)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  5,  6,
  GROUP 1 RANGE (   7-   7)     GROUP 2 RANGE (   8-   9)     GROUP 1 STEPSIZE (   0.000)     GROUP 2 STEPSIZE (   1.000)

  GROUP 1 COLUMN(S) =  3,
  GROUP 2 COLUMN(S) =  4,  5,  6,
  GROUP 1 RANGE (   8-   8)     GROUP 2 RANGE (  13-  14)     GROUP 1 STEPSIZE (   0.000)     GROUP 2 STEPSIZE (   1.000)
GROUP 1 COLUMN(S) =  3,
GROUP 2 COLUMN(S) =  4,  5,  6,
GROUP 1 RANGE (  7-   7)       GROUP 2 RANGE ( 14-  15)       GROUP 1 STEPSIZE (   0.000)       GROUP 2 STEPSIZE (   1.000)

GROUP 1 COLUMN(S) =  1,  2,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  4-  13)       GROUP 2 RANGE (  9-   7)       GROUP 1 STEPSIZE (   1.286)       GROUP 2 STEPSIZE (  -0.286)

GROUP 1 COLUMN(S) =  1,  2,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE ( 12-  18)       GROUP 2 RANGE (  7-   2)       GROUP 1 STEPSIZE (   1.500)       GROUP 2 STEPSIZE (  -1.250)

GROUP 1 COLUMN(S) =  1,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  3-  17)       GROUP 2 RANGE (  9-   4)       GROUP 1 STEPSIZE (   1.273)       GROUP 2 STEPSIZE (  -0.455)

GROUP 1 COLUMN(S) =  1,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  7-  10)       GROUP 2 RANGE (  8-   7)       GROUP 1 STEPSIZE (   1.000)       GROUP 2 STEPSIZE (  -0.333)

GROUP 1 COLUMN(S) =  1,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  2-  13)       GROUP 2 RANGE (  9-   8)       GROUP 1 STEPSIZE (   1.833)       GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  1,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE ( 13-  18)       GROUP 2 RANGE (  7-   2)       GROUP 1 STEPSIZE (   1.000)       GROUP 2 STEPSIZE (  -1.000)

GROUP 1 COLUMN(S) =  2,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  5-  12)       GROUP 2 RANGE (  9-   7)       GROUP 1 STEPSIZE (   1.000)       GROUP 2 STEPSIZE (  -0.286)

GROUP 1 COLUMN(S) =  2,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  7-   8)       GROUP 2 RANGE (  8-   7)       GROUP 1 STEPSIZE (   0.500)       GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  4-  12)       GROUP 2 RANGE (  9-   8)       GROUP 1 STEPSIZE (   1.333)       GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  1,  2,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  6-  19)       GROUP 2 RANGE (  9-   7)       GROUP 1 STEPSIZE (   1.857)       GROUP 2 STEPSIZE (  -0.286)

GROUP 1 COLUMN(S) =  1,  2,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE ( 12-  15)       GROUP 2 RANGE (  8-   7)       GROUP 1 STEPSIZE (   1.500)       GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  2,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  5-  18)       GROUP 2 RANGE (  9-   8)       GROUP 1 STEPSIZE (   2.167)       GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  1,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE ( 12-  15)       GROUP 2 RANGE (  8-   7)       GROUP 1 STEPSIZE (   1.500)       GROUP 2 STEPSIZE (  -0.500)
GROUP 1 COLUMN(S) =  1,  4,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   4-  18)      GROUP 2 RANGE (   9-   8)      GROUP 1 STEPSIZE (    2.333)      GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  1,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  13-  15)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    2.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  2,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  12-  14)      GROUP 2 RANGE (   8-   7)      GROUP 1 STEPSIZE (    1.000)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,  4,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   6-  17)      GROUP 2 RANGE (   9-   8)      GROUP 1 STEPSIZE (    1.833)      GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  2,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  13-  14)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    1.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  1,  2,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  17-  21)      GROUP 2 RANGE (   8-   7)      GROUP 1 STEPSIZE (    2.000)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  2,  4,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   7-  23)      GROUP 2 RANGE (   9-   8)      GROUP 1 STEPSIZE (    2.667)      GROUP 2 STEPSIZE (  -0.167)

GROUP 1 COLUMN(S) =  1,  2,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  18-  20)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    2.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  1,  4,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  18-  20)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    2.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  2,  4,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  18-  19)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    1.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  1,  2,  4,  5,  6,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  23-  25)      GROUP 2 RANGE (   8-   8)      GROUP 1 STEPSIZE (    2.000)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  1,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   4-   7)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (    0.750)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   4-   8)      GROUP 2 RANGE ( [illegible]-   6)      GROUP 1 STEPSIZE (    1.000)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  3,
GROUP 2 COLUMN(S) =  4,
GROUP 1 RANGE (   8-   6)      GROUP 2 RANGE (   5-   7)      GROUP 1 STEPSIZE (  -0.500)      GROUP 2 STEPSIZE (   0.500)
GROUP 1 COLUMN(S) =  3,
GROUP 2 COLUMN(S) =  5,
GROUP 1 RANGE (   8-   6)      GROUP 2 RANGE (   2-   2)      GROUP 1 STEPSIZE (  -0.500)      GROUP 2 STEPSIZE (   0.000)

GROUP 1 COLUMN(S) =  3,
GROUP 2 COLUMN(S) =  4,  5,
GROUP 1 RANGE (   8-   6)      GROUP 2 RANGE (   7-   9)      GROUP 1 STEPSIZE (  -0.500)      GROUP 2 STEPSIZE (   0.500)

GROUP 1 COLUMN(S) =  1,  2,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   8-  15)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.750)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   9-  14)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.250)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   6-   9)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   0.750)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   9-  15)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.500)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (   6-  10)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.000)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  2,  4,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  13-  22)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   2.250)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  2,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  10-  17)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.750)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  11-  16)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.250)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  2,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  11-  17)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   1.500)      GROUP 2 STEPSIZE (  -0.500)

GROUP 1 COLUMN(S) =  1,  2,  4,  5,
GROUP 2 COLUMN(S) =  3,
GROUP 1 RANGE (  15-  24)      GROUP 2 RANGE (   8-   6)      GROUP 1 STEPSIZE (   2.250)      GROUP 2 STEPSIZE (  -0.500)
```

APPENDIX V

MATRIX INSTRUCTION MATERIALS

CHAPTER 1

Definition, Equality, Addition, and Subtraction of Matrices

A *matrix* is a rectangular array of elements. If a matrix has n rows and m columns, it is said to be an *n x m* (read *n* by *m*) matrix. If a matrix happens to have the same number of rows as columns it is called a *square matrix*. All matrices to be considered in this program will be square matrices. Unless we say otherwise, all elements of the matrices in this program will be *real numbers*. The real numbers include: (1) *the integers*: 0, ± 1, ± 2, ... , (2) *the rational numbers*: that is, all numbers which *can* be expressed as the *ratio* of two integers: e.g.; 1/2, - 4/31, 15/7, etc., and (3) *the irrational numbers*: that is, all numbers which *cannot* be expressed as the *ratio* of two integers: e.g.; $\sqrt{2}$, π, $-e^5$, etc. A matrix, then, is an arrangement or array of real numbers in a square.

A matrix having n rows and n columns is an n x n matrix, or a matrix of size n.

Example:

$$\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1.4 & 2\pi \\ -5/6 & 1/8 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 516 & 3.2 \end{bmatrix}$$

are 2 x 2 matrices, or matrices of size 2,

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & e^3 & 3/4 \\ 0 & -7 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6.2 & 5 & 3 \\ 0 & 0 & 1 \\ 4.2 & -61 & 8642 \end{bmatrix} \quad \text{are}$$

3 x 3 matrices, or matrices of size 3, and

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 3 & 0 & 4 \\ 3 & 6 & 6 & -5 \\ 1 & 1 & 4 & 3 \end{bmatrix} \quad \text{is a matrix of size 4.}$$

The main advantage to writing an array of numbers as we have done is that it is a convenient way to treat many numbers as a single object. Thus, we might call our first matrix A, our second B, etc.

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.4 & 2\pi \\ -5/6 & 1/8 \end{bmatrix}, \quad \text{etc.}$$

Definition 1.1 : Let A be a matrix. Then the symbol $[A]_{i,j}$ will denote the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of A.

Example: If K is the 7 x 7 matrix

$$K = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 10 & 9 \\ 5 & 4 & -3 & -4 & -5 & 15 & 13 \\ 0 & 6 & 7 & 8 & -2 & 14 & 12 \\ -2 & -3 & -1 & -6 & -8 & -9 & -7 \\ 17 & 43 & -10 & 11 & -11 & -45 & 16 \\ 19 & 18 & -12 & 84 & 63 & 22 & 27 \\ 83 & 42 & 77 & 91 & 136 & -61 & -19 \end{bmatrix}, \text{ the element}$$

in the fourth row and fifth column is -8; that is, $[K]_{4,5} = -8$.

The element in the seventh row and second column is 42; $[K]_{7,2} = 42$. In this matrix it happens that $[K]_{1,2} = [K]_{1,4}$ and $[K]_{2,3} = [K]_{4,2}$.

Problem Section 1.1

1. Let $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (a) What is $[I]_{4,4}$? What is $[I]_{3,4}$? $[I]_{4,3}$?

   (b) Write down the elements of row three. List the elements of column three. What element is in both row three and column three?

   (c) When is $[I]_{i,j} \neq 0$ ("$\neq$" read "not equal")? When is $[I]_{i,j} = 0$?

2. (a) Write a 2 x 2 matrix A such that $[A]_{2,2} = 0$ and all other elements of A are greater than the number 4.

   (b) How many elements are there in a 2 x 2 matrix? In a 3 x 3 matrix? In a 4 x 4 matrix? In an n x n matrix?

Definition 1.2: Two matrices A and B are said to be equal if

(1) They are of the same size, and

(2) Each element of A is equal to the corresponding element of B.

Example: Let A be an m x m matrix, let B be an n x n matrix, and suppose $m \neq n$. Then A cannot possible be equal to B, regardless of what the elements of A and B happen to be. Now assume $m = n$. Then $A = B$ if and only if* $[A]_{i,j} = [B]_{i,j}$ for all i and j between 1 and n. Thus if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 2 \\ 0 & 3 & -1 \end{bmatrix} \qquad \text{and } B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 2 \\ 0 & 3 & -1 \end{bmatrix},$$

then $A = B$, but if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 2 \\ 0 & 3 & -1 \end{bmatrix} \qquad \text{and } B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 2 \\ 0 & 2 & -1 \end{bmatrix},$$

then $A \neq B$.

This definition of equality of matrices enables us to express certain relationships more compactly.

Rather than writing

$$\begin{aligned} 3x + 2y &= 7 \\ x - y &= 1 \\ k + h &= 4 \\ k + 5h &= 15, \end{aligned}$$

we can write

$$\begin{bmatrix} 3x + 2y & k + h \\ x - y & k + 5h \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 1 & 15 \end{bmatrix}.$$

*By "x if and only if y" we mean both (1) if x then y, and (2) if y then x.

A matrix equation as innocuous looking as

$$D = C$$

might represent

$$\begin{bmatrix} a + a & a + b & a + c \\ b + a & b + b & b + c \\ c + a & c + b & c + c \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix},$$

which in turn is a systematic representation of the following nine equations:

$a + a = 2$, $a + b = 3$, $a + c = 4$,

$b + a = 3$, $b + b = 4$, $b + c = 5$,

$c + a = 4$, $c + b = 5$, $c + c = 6$.

It should be clear that to know a matrix is to know all of its elements. Thus a matrix is determined when all of its elements have been specified.

Example: A 2 x 2 matrix A is determined when we specify that $[A]_{1,1} = 2\pi$, $[A]_{1,2} = \sqrt[3]{5}$, $[A]_{2,1} = \sqrt{2}$, and $[A]_{2,2} = 17$. In this case,

$$A = \begin{bmatrix} 2\pi & \sqrt[3]{5} \\ \sqrt{2} & 17 \end{bmatrix}.$$

Suppose

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \text{ then}$$

B is determined. Another way to determine B is to specify that

$$[B]_{i,j} = 0 \text{ when } i > j$$

and

$$[B]_{i,j} = 1 \text{ when } i \leq j.$$

The zero matrix is a matrix all of whose elements are the number 0 (we shall print 0 when we mean the number zero, and $\mathcal{O}$ when we mean the zero matrix). The 2 x 2 zero matrix, which is denoted by $\mathcal{O}_2$, is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, $\mathcal{O}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc.

Thus the symbol $\mathcal{O}_n$ means the n x n zero matrix. Since one can always tell from context what size zero matrix is being considered, the subscript will be excluded. That is, all zero matrices, regardless of size, will be written as $\mathcal{O}$.

Problem Section 1.2

1. Use the definition of equality of matrices to find x, y, a, and b in the following matrix equation. Be sure to check your work by substituting the numbers you obtain back into the matrix equation.

The equation is $A = B$,

$$\text{where } A = \begin{bmatrix} x + 2y & a - 3b \\ -x-y & 2a + 4b \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 0 & 1 \\ 1/2 & 62 \end{bmatrix}.$$

2. Specify a 4×4 matrix whose elements are all negative numbers.

3. Specify a 5×5 matrix having all non-negative numbers for elements.

4. Specify a non-zero 3×3 matrix A such that $[A]_{i,j} = [A]_{j,i}$.

5. (a) Specify another non-zero 3×3 matrix B such that $[B]_{i,j} = -[B]_{j,i}$.

   (b) If B is such a matrix what must $[B]_{1,1}$ be? How about $[B]_{2,2}$ and $[B]_{3,3}$?

To make the study of matrices meaningful and useful we must now consider basic operations with matrices. We will begin by defining "sum" for matrices. "Product" will be considered in chapter 2.

Definition 1.3: Let A and B be two $n \times n$ matrices. The sum of A and B — i.e., $A + B$ — is the $n \times n$ matrix C whose element in the i^{th} row and j^{th} column is the sum of $[A]_{i,j}$ and $[B]_{i,j}$. That is $[C]_{i,j} = [A + B]_{i,j} = [A]_{i,j} + [B]_{i,j}$.

Thus matrices of the same size are added by adding each element of one matrix to the corresponding element of the other matrix.

Example:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+2 \\ 4+1 & 5+0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1-\sqrt{3} & 4 \\ 28 & -\sqrt{5} \end{bmatrix} + \begin{bmatrix} -1+\sqrt{3} & 7 \\ -19 & -2\sqrt{5} \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ -19 & 4\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -10 & \sqrt{5} \end{bmatrix}$$

Matrices of different sizes are not to be added. That is, we shall not add an n x n matrix to an m x m matrix unless m = n.

Problem Section 1.3

1. Find the following sums:

(a) $$\begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 4/5 \end{bmatrix} + \begin{bmatrix} 1/2 & 2/4 \\ 5/6 & 1/5 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

(b) $$\begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix} + \begin{bmatrix} 1-a & -b & -c \\ -x & 1-y & -z \\ -u & -v & 1-w \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

2. Does the sum

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

make sense?

3. What about the sum

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ?$$

Find this latter sum.

We have already defined the sum of two n x n matrices A and B as the n x n matrix C whose elements are $[C]_{i,j} = [A]_{i,j} + [B]_{i,j}$. In a similar way, we can define D as the sum B + A, the elements of D being $[D]_{i,j} = [B]_{i,j} + [A]_{i,j}$. Since the elements of a matrix are numbers and since the addition of numbers is commutative — e.g., 2 + 3 = 3 + 2, or, more generally, $x + y = y + x$ — $[C]_{i,j} = [A + B]_{i,j} = [A]_{i,j} + [B]_{i,j}$

= B

$$= [B]_{i,j} + [A]_{i,j}$$

$$= [B + A]_{i,j} = [D]_{i,j}.$$

In short, $[C]_{i,j} = [D]_{i,j}$; therefore, C = D.

It follows from this that A + B = B + A. This proves

Theorem 1.1: Let A and B be two n x n matrices. Then, A + B = B + A.

We have proved this theorem by making use of something we already know about numbers. This was possible because: (1) the elements of matrices are numbers, (2) the addition of two matrices is defined as the addition of their corresponding elements, and (3) the addition of numbers is commutative. We shall soon see that each of the basic laws for the addition of numbers has an analogous matric counterpart. For this reason, familiar number properties will be used to develop corresponding matrix properties. An immediate example of this approach is illustrated in

Theorem 1.2: Let A, B, and C be n x n matrices.

Then $(A + B) + C = A + (B + C)$

Proof: Once again we will use the notation for elements in a matrix.

$$\begin{aligned} [(A + B) + C]_{i,j} &= [A + B]_{i,j} + [C]_{i,j} \\ &= [A]_{i,j} + [B]_{i,j} + [C]_{i,j} \\ &= [A]_{i,j} + [B + C]_{i,j} \\ &= [A + (B + C)]_{i,j}. \end{aligned}$$

Now every element of $(A + B) + C$ is equal to the corresponding element of $A + (B + C)$ and both sums are n x n matrices (are of the same size), so $(A + B) + C = A + (B + C)$. Notice that in the above proof we were working with the elements of matrices. These elements are numbers and numbers are associative under addition; e.g., $1 + 2 + 3 = (1 + 2) + 3 = 1 + (2 + 3)$, or, more generally, $x + y + z = (x + y) + z = x + (y + z)$. Note that by $(x + y) + z$ we mean first add y to x and then add z to the result. Since it has been shown that the way in which matrices are grouped (associated) in addition is immaterial, unnecessary parentheses shall always be omitted.

Theorem 1.3: Let A be an n x n matrix. Then $A + \bigcirc = \bigcirc + A = A$.

Proof: This one is left to the reader. Here, of course, $\bigcirc$ is the n x n zero matrix.

Once we know now to add numbers, it is customary to consider multiplication of numbers. For example, $2 + 2 = 2 \cdot 2$, $2 + 2 + 2 = 3 \cdot 2$, etc. Similarly for matrices, $A + A = 2 \cdot A$, $A + A + A = 3 \cdot A$, etc. The matrix $3A$ is clearly the matrix whose elements are each thrice those of the corresponding elements of A.

Example:

Suppose $A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$.

$$\text{Then } 3A = A + A + A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 3 & 3 \cdot 6 \\ 3 \cdot 9 & 3 \cdot 12 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 27 & 36 \end{bmatrix}.$$

The number preceding A need not be an integer. Consider, for instance,

$$A = 1/3A + 1/3A + 1/3A.$$

$$A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} (1 + 1 + 1) & (2 + 2 + 2) \\ (3 + 3 + 3) & (4 + 4 + 4) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= 1/3A + 1/3A + 1/3A.$$

These examples involving numerical multiples of a matrix lead us to

Definition 1.4: Let A be an $n \times n$ matrix and let x be a number. Then the matrix whose element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column is $[xA]_{i,j}$ is the product of A by the number x. More concisely, xA is defined by $[xA]_{i,j} = x[A]_{i,j}$

Notice that the product of a matrix by a number is another matrix. It is also possible to define the product of two matrices, and we shall do so in chapter 2. The following theorem deals with facts about numerical multiples of matrices.

Theorem 1.4: Let A and B be matrices of the same size and let x and y be numbers; then

(1) $x(yA) = (xy)A$

(2) $(x + y)A = (xA) + (yA)$

(3) $(-1)A = -A$

(4) $x(A+B) = (xA) + (xB)$

(5) $x \cdot \mathbf{O} = \mathbf{O}$

(6) $0 \cdot A = \mathbf{O}$

Proof of (1): Using our notation for the elements in a matrix:

$$\begin{aligned}[x(yA)]_{i,j} = x[yA]_{i,j} &= x\{y[A]_{i,j}\} \\ &= (xy)[A]_{i,j} \\ &= [(xy)A]_{i,j}\end{aligned}$$

Now we shall express the same proof verbally.
The element in the i^{th} row and j^{th} column of yA is, by definition, $y[A]_{i,j}$. Therefore, the element in the i^{th} row and j^{th} column of $x(yA)$ is $x\{y[A]_{i,j}\}$. Now the elements in a matrix are numbers and numbers obey the associative law of multiplication; e.g., $2(3 \cdot 4) = (2 \cdot 3)4$, or, more generally, $x(y \cdot z) = (x \cdot y)z$. Using this idea we have $x\{y[A]_{i,j}\} = (xy)[A]_{i,j}$. It follows from the definition of the product of a matrix by a

number that $(xy)\,[A]_{i,j} = [(xy)\,A]_{i,j}$. But $[(xy)\,A]_{i,j}$ is, by definition, the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of $(xy)A$. Thus each element of the matrix $x(yA)$ is equal to the corresponding element of the matrix $(xy)A$. Therefore, $x(yA) = (xy)A$.

Proof of (4): The more concise, symbolic proof is:

$$\begin{aligned}
[x(A+B)]_{i,j} = x\,[A+B]_{i,j} &= x\left\{[A]_{i,j} + [B]_{i,j}\right\} \\
&= x\,[A]_{i,j} + x\,[B]_{i,j} \\
&= [xA]_{i,j} + [xB]_{i,j} \\
&= [xA + xB]_{i,j}
\end{aligned}$$

The corresponding verbal argument is as follows: The element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of the matrix $A + B$ is $[A+B]_{i,j}$; therefore, the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of $x\,(A+B)$ is $[x(A+B)]_{i,j}$. By definition 1.4,

$$\begin{aligned}
[x(A+B)]_{i,j} &= x\,[A+B]_{i,j} \\
&= x\left\{[A]_{i,j} + [B]_{i,j}\right\}. \quad \text{(definition of addition of matrices)}
\end{aligned}$$

Now x, $[A]_{i,j}$, and $[B]_{i,j}$ are all numbers, and for numbers the distributive law holds; e.g., $2(3 + 4) = 2\cdot 3 + 2\cdot 4$, or, in general, $x(y + z) = x\cdot y + x\cdot z$, therefore,

$$\begin{aligned}
x\left\{[A]_{i,j} + [B]_{i,j}\right\} &= x\,[A]_{i,j} + x\,[B]_{i,j} \\
&= [xA]_{i,j} + [xB]_{i,j} \quad \text{(definition 1.4)} \\
&= [xA + xB]_{i,j} \quad \text{(definition of addition of matrices)}
\end{aligned}$$

This last expression is the element in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column of the matrix xA + xB. It follows from this that every element in the matrix x(A + B) is equal to the corresponding element in the matrix xA + xB, or x(A + B) = xA + xB.

Our definition of xA was given by $[xA]_{i,j} = x\,[A]_{i,j}$. When x = -1 we have part (3) of theorem 1.4. This means that to multiply a matrix by -1, one simply multiplies each element by -1.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and let x = -1.

Then $xA = (-1)A = -A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$.

Thus the negative of A is, as we might have suspected, -A.

Definition 1.5: Let A and B be matrices of the same size. Then subtraction of B from A is the same as addition of -B to A. That is A-B is defined as A + (-B).

Example:

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & -3 \\ -8 & -2 & 1 \\ 4 & 4 & -4 \end{bmatrix}$

Then $A - B = \begin{bmatrix} 3 & 1 & 5 \\ 8 & 1 & 3 \\ -3 & -3 & 5 \end{bmatrix}$.

Also, $B - A = \begin{bmatrix} -3 & -1 & -5 \\ -8 & -1 & -3 \\ 3 & 3 & -5 \end{bmatrix}$

Note that $A - B = -(B - A)$.

Theorem 1.5: If A and B are matrices of the same size, then:

(1) $A + (-A) = O$

(2) $-(-A) = A$

(3) $-O = O$

(4) $-(A + B) = (-A) + (-B)$

Proof of (1): The matrix xA has been defined by

$[xA]_{i,j} = x\,[A]_{i,j}$. When $x = -1$, $(-1)A = -A$ is defined as $[-A]_{i,j} = -[A]_{i,j}$.

Thus

$$\begin{aligned} [A + (-A)]_{i,j} &= [A]_{i,j} + [-A]_{i,j} \\ &= [A]_{i,j} - [A]_{i,j} \\ &= 0 \end{aligned}$$

So $A + (-A) = O$

The corresponding verbal proof is as follows: The element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of $-A$ is, by definition, $-[A]_{i,j}$. Therefore, the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of $A + (-A)$ is $[A]_{i,j} + \{-[A]_{i,j}\}$. But these latter are merely numbers, so the sum is zero. This proves that every element of $A + (-A)$ is zero. Then $A + (-A)$ is a matrix all of whose elements are zero; viz., the O matrix.

Problem Section 1.4

1. Prove (2) of theorem 1.5.
2. Prove (4) of theorem 1.5.
3. Prove (2) of theorem 1.4.
4. Prove (6) of theorem 1.4.
5. Compute

(a) $$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 5 & -8 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1/2 & 1/8 & 3/8 \\ 1/4 & 1/3 & 1/5 \\ 5/4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} .50 & .25 & .50 \\ .75 & 0 & .80 \\ 2.50 & 4 & 3 \end{bmatrix}$$

(c) $$3 \begin{bmatrix} 2 & 2 \\ 1 & 5 \end{bmatrix} + 1/2 \begin{bmatrix} 8 & 4 \\ 2 & 4 \end{bmatrix}$$

(d) $$10 \begin{bmatrix} .30 & .40 \\ .80 & 2.50 \end{bmatrix} - 15 \begin{bmatrix} 8/3 & 4/15 \\ 8/15 & 17/15 \end{bmatrix}$$

(answer: $\begin{bmatrix} -37 & 0 \\ 0 & 8 \end{bmatrix}$

Suppose A and B are known matrices of the same size. Now consider the matrix equation X + A = B, where X is of the same size as A and B but is unknown. If this equation involved ordinary numbers rather than matrices we would simply add -A to both sides of the equation; i.e.,

$$X + A + (-A) = B + (-A)$$

$$X = B - A.$$

It happens that we can do the very same sort of thing with matrices.

Example: Given $Y + C = D$,

where $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, find Y.

$$Y + C = D$$

$$Y = D - C$$

$$Y = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix}$$

Just as in ordinary algebra, we can check the solution by substituting Y back into the original equation.

The check goes as follows:

$Y + C \overset{?}{=} D$ ("$\overset{?}{=}$" read "does it equal?")

$$\begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Sometimes the equations are a bit more complicated, but the procedure is still the same as the one used for the algebra of numbers.

Example:

Solve $-3/4X ++13A + 1/2\,B = -3C + 9D$ for the matrix X, given

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & -9 \\ 6 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}, \quad \text{and } D = \begin{bmatrix} -1/2 & -1/4 \\ 3/4 & 11/12 \end{bmatrix}$$

Multiplying the equation through by $-4/3$ gives

$$X - 52/3\,A - 2/3\,B = 4C - 12D$$

Adding $52/3\,A + 2/3\,B$ to both sides of the equation gives

$$X = 52/3\,A + 2/3\,B + 4C - 12D\,.$$

At this point we make use of the given information.

$$X = 52/3 \begin{bmatrix} 0 & 0 \\ 0 & -9 \end{bmatrix} + 2/3 \begin{bmatrix} 15 & -9 \\ 6 & 0 \end{bmatrix} + 4 \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} - 12 \begin{bmatrix} -1/2 & -1/4 \\ 3/4 & 11/12 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 \\ 0 & -156 \end{bmatrix} + \begin{bmatrix} 10 & -6 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ 4 & 16 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ -9 & -11 \end{bmatrix}$$

$$X = \begin{bmatrix} 24 & 13 \\ -1 & -151 \end{bmatrix}$$

Problem Section 1.5

1. Solve the equation

$$X + \begin{bmatrix} 3 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

for the matrix X.

2. Solve the equation

$$Z - \begin{bmatrix} 1/2 & 1/4 \\ 1/6 & 1/8 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/4 \\ -1/6 & -1/8 \end{bmatrix}$$

for the matrix Z.

3. Solve the equation

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - W + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 9 & 8 \end{bmatrix}$$

for the matrix W.

4. Solve the following equation for X and check your answer by substituting the matrix you obtain back into the given equation.

$$4\left(3X - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3/2\,X\right) = 10\,X + 60 \begin{bmatrix} 1/2 & 1/3 \\ 1/4 & 1/5 \end{bmatrix}$$

5. Given

$$A = \begin{bmatrix} 1/2 & 1/10 & 1/5 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 7 \\ 0 & 0 & 4 \\ -1 & -1 & -2 \end{bmatrix} \quad \text{and}$$

$$C = \begin{bmatrix} 11 & 2 & 1 \\ 1 & 0 & 0 \\ -5 & 6 & 7 \end{bmatrix},$$

find the matrix Z in the following equation:

$$10(Z + A) - 15(Z + B) + 4(Z + C) = 0$$

(Your solution should be $Z = \begin{bmatrix} 4 & -6 & -99 \\ 24 & 10 & -50 \\ -15 & 39 & 48 \end{bmatrix}$

6. Solve the simultaneous system:

$$X + 5Y = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

$$-3X + \tfrac{1}{2}Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for the matrices X and Y.

CHAPTER 2

Multiplication of Matrices

So far we have defined and studied:

(1) Addition of matrices

(2) Multiplication of a matrix by a number

(3) Subtraction of matrices.

In this chapter we shall define the product of two matrices. We shall use a simple practical problem to lead us intuitively into a means of combining two matrices which will later be defined, in a more formal way, as multiplication.

Suppose that a school superintendent is an administrator for a county which has four public schools. The following table lists the schools and the educational preparation of the teachers.

Table 2.1

Schools	No degree	B.A.	M.A.	M.A. + 30 credits
Essex	0	15	3	0
Heavenport	1	9	1	0
Kingsley	1	16	2	1
Shorebrook	0	20	10	1

Table 2.1 can also be arranged in the form of a matrix.

$$\begin{array}{cc} & \overbrace{\begin{array}{cccc}\text{N.D.} & \text{B.A.} & \text{M.A.} & \text{M.A. + 30 credits}\end{array}}^{\text{Educational Preparation of Teachers}} \\ \text{Schools}\left\{\vphantom{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}\right. & \begin{bmatrix}0 & 15 & 3 & 0\\ 1 & 9 & 1 & 0\\ 1 & 16 & 2 & 1\\ 0 & 20 & 10 & 1\end{bmatrix}\end{array}$$

The base pay schedule for the county is presented in Table 2.2.

Table 2.2

Educational Preparation	Base Pay
N.D.	$3500
B.A.	4000
M.A.	4500
M.A. + 30 credits	5000

Table 2.2 can also be written as a matrix.

$$\begin{array}{cl} & \text{Base Pay} \\ \text{Educational Preparation} \left\{ \begin{array}{l} \text{N.D.} \\ \text{B.A.} \\ \text{M.A.} \\ \text{M.A.} + \end{array} \right. & \begin{bmatrix} 3500 & 0 & 0 & 0 \\ 4000 & 0 & 0 & 0 \\ 4500 & 0 & 0 & 0 \\ 5000 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

A matrix such as this latter one is called a vector. Vectors shall be discussed in chapter 4.

The first of the above matrices might be called the school-educational preparation matrix. The second might be called the educational preparation - base pay scale matrix. It should be clear that by combining these two sets of information the superintendent can answer certain questions concerning his payroll. For example, the total base pay for the Essex school is computed as follows:

0 teachers having no degree, each receiving $3500 = $ 0

15 " " B.A., " " $4000 = 60,000

3 " " M.A., " " $4500 = 13,500

0 " " M.A. +, " " $5000 = 0

Total $ 73,500

To obtain this total, we multiplied each element in the Essex School row of the school-educational preparation matrix (moving from left to right) by each corresponding column element of the first column of the educational preparation - base pay scale matrix (moving from top to bottom) and then added the resulting products. This procedure, of course, is perfectly general. Thus, to find the total base pay for Kingsley School, we multiply each element in the Kingsley School row of the first matrix by the corresponding element of the first column of the second matrix and then add the four products; i.e., (1 x 3500) + (16 x 4000) + (2 x 4500) + (1 x 5000) = $81,500. The superintendent could complete the process as follows.

$$\begin{bmatrix} 0 & 15 & 3 & 0 \\ 1 & 9 & 1 & 0 \\ 1 & 16 & 2 & 1 \\ 0 & 20 & 10 & 1 \end{bmatrix} \begin{bmatrix} 3500 & 0 & 0 & 0 \\ 4000 & 0 & 0 & 0 \\ 4500 & 0 & 0 & 0 \\ 5000 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix}(0 \times 3500) + (15 \times 4000) \\ + (3 \times 4500) + (0 \times 5000)\end{matrix} & \cdot & \cdot & \begin{matrix}(0 \times 0) + (15 \times 0) \\ + (3 \times 0) + (0 \times 0)\end{matrix} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \begin{matrix}(0 \times 3500) + (20 \times 4000) \\ + (10 \times 4500) + (1 \times 5000)\end{matrix} & & & \begin{matrix}(0 \times 0) + (20 \times 0) \\ + (10 \times 0) + (1 \times 0)\end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} 73500 & 0 & 0 & 0 \\ 44000 & 0 & 0 & 0 \\ 81500 & 0 & 0 & 0 \\ 130000 & 0 & 0 & 0 \end{bmatrix}$$

We shall now consider that part of the payroll determined by educational experience. The following matrix shows the educational experience of the teachers at the four schools.

Schools	Educational Experience			
	Less than 5 years	5 - 10 years	10 - 15 years	15 + years
Essex	10	5	2	1
Heavenport	4	6	1	0
Kingsley	8	10	0	2
Shorebrook	15	7	8	1

Pay increments for teaching experience is given by the following matrix:

Pay Increment

$$\begin{array}{l} \text{Less than 5 years} \\ \text{5 - 10 years} \\ \text{10 - 15 years} \\ \text{More than 15 years} \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 600 & 0 & 0 & 0 \\ 1200 & 0 & 0 & 0 \\ 1800 & 0 & 0 & 0 \end{bmatrix}$$

Combining these latter two matrices in the same way the former two were combined gives a matrix which accounts for that portion of the payroll which depends upon teaching experience.

$$\begin{bmatrix} 10 & 5 & 2 & 1 \\ 4 & 6 & 1 & 0 \\ 8 & 10 & 0 & 2 \\ 15 & 7 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 600 & 0 & 0 & 0 \\ 1200 & 0 & 0 & 0 \\ 1800 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7200 & 0 & 0 & 0 \\ 4800 & 0 & 0 & 0 \\ 9600 & 0 & 0 & 0 \\ 15600 & 0 & 0 & 0 \end{bmatrix}$$

The result might be called the school - pay increment matrix. By adding the school - pay increment matrix to the school - base pay scale matrix the superintendent determines his payroll for each school for the current year.

$$\begin{bmatrix} \underset{73500}{\text{[illegible]}} & 0 & 0 & 0 \\ 44000 & 0 & 0 & 0 \\ 81500 & 0 & 0 & 0 \\ 130000 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 7200 & 0 & 0 & 0 \\ 4800 & 0 & 0 & 0 \\ 9600 & 0 & 0 & 0 \\ 15600 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \overset{80700}{\text{[illegible]}} & 0 & 0 & 0 \\ 48800 & 0 & 0 & 0 \\ 91100 & 0 & 0 & 0 \\ 145600 & 0 & 0 & 0 \end{bmatrix}$$

He can also determine the total payroll for the teachers of his county by totaling the numbers in the first column.

Essex payroll	80,700 $ [illegible]
Heavenport payroll	48,800
Kingsley payroll	91,100
Shoreport payroll	145,600
Total	366,200 [illegible]

We shall now apply the notation for the elements of a matrix to the combination problem which we have been considering. Let A and B be two 4 x 4 matrices. Then the element in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column of the matrix AB will be

$$[A]_{i,1}\,[B]_{1,j} + [A]_{i,2}\,[B]_{2,j} + [A]_{i,3}\,[B]_{3,j} + [A]_{i,4}\,[B]_{4,j}.$$

This mathematical rule for combining matrices is known as matrix multiplication. The rule is general so we need not restrict ourselves to 4 x 4 matrices.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 11 & -4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ 5 & -1 & -5 \\ 8 & -1 & -8 \end{bmatrix}$$

We shall now give a formal definition of the product of two matrices.

Definition 2.1: Let A and B be two n x n matrices. Then the product of A and B—i.e., AB— is the matrix whose element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column is

$$[AB]_{i,j} = [A]_{i,1}\ [B]_{1,j} + [A]_{i,2}\ [B]_{2,j} + \ldots + [A]_{i,n}\ [B]_{n,j} .$$

Although this definition may seem rather formidable at first glance, we shall soon see that it is very useful and quite straightforward. Let us first consider the above definition in words. We find the element in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column of AB as follows:

Multiply the first entry in row i of A by the first entry in column j of B.

Multiply the second entry in row i of A by the second entry in column j of B.

This process is continued until finally we:

Multiply the n^{th} entry in row i of A by the n^{th} entry in column j of B.

Finally, add all of the above products together.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Then

$$[AB]_{1,1} = [A]_{1,1}\ [B]_{1,1} + [A]_{1,2}\ [B]_{2,1}$$
$$= (1 \times 1) + (0 \times 1) = 1$$

$$[AB]_{1,2} = [A]_{1,1}\ [B]_{1,2} + [A]_{1,2}\ [B]_{2,2}$$
$$= (1 \times 2) + (0 \times 1) = 2$$

$$[AB]_{2,1} = [A]_{2,1}\,[B]_{1,1} + [A]_{2,2}\,[B]_{2,1}$$

$$= (2 \times 1) + (4 \times 1)$$

$$= 6$$

$$[AB]_{2,2} = [A]_{2,1}\,[B]_{1,2} + [A]_{2,2}\,[B]_{2,2}$$

$$= (2 \times 2) + (4 \times 1)$$

$$= 8$$

So the product, AB, is $\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}$

The definition of the product of two matrices can also be expressed in summation notation. Let A and B two n x n matrices and define Σ as follows:

$$\sum_{k=1}^{n} Xk = X_1 + X_2 + \dots + X_n .$$

Then the sum $[A]_{i,1}\,[B]_{1,j} + [A]_{i,2}\,[B]_{2,j} + \dots + [A]_{i,n}\,[B]_{n,j}$ is more compactly expressed as

$$\sum_{k=1}^{n} [A]_{i,k}\,[B]_{k,j} .$$

That is,

$$[AB]_{i,j} = \sum_{k=1}^{n} [A]_{i,k}\,[B]_{k,j} .$$

Remember that matrix product has only been defined for matrices of the same size. We will not even consider the product of matrices of different sizes. Further illustrations shall now be given to help clarify the concept of matrix multiplication.

Example: Given $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 2/3 \\ 4 & -1 \end{bmatrix}$

Find AB and BA.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1/2 & 2/3 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1/2 + 2 \times 4 & 1 \times 2/3 + 2 \times -1 \\ 3 \times 1/2 + -2 \times 4 & 3 \times 2/3 + -2 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 17/2 & -4/3 \\ -13/2 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1/2 & 2/3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \times 1 + 2/3 \times 3 & 1/2 \times 2 + 2/3 \times -2 \\ 4 \times 1 + -1 \times 3 & 4 \times 2 + -1 \times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 & -1/3 \\ 1 & 10 \end{bmatrix}$$

Given

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } D = \begin{bmatrix} x & 2y \\ z & -w \end{bmatrix},$$

find CC, DD, CD, and DC.

$$CC = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$$

$$DD = \begin{bmatrix} x & 2y \\ z & -w \end{bmatrix} \begin{bmatrix} x & 2y \\ z & -w \end{bmatrix} = \begin{bmatrix} x^2 + 2yz & 2xy - 2wy \\ xz - wz & 2yz + w^2 \end{bmatrix}$$

$$CD = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & 2y \\ z & -w \end{bmatrix} = \begin{bmatrix} ax + bz & 2ay - bw \\ cx + dz & 2cy - dw \end{bmatrix}$$

$$DC = \begin{bmatrix} x & 2y \\ z & -w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax + 2cy & bx + 2dy \\ az - cw & bz - dw \end{bmatrix}$$

Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 & 2 & 4 \\ 3 & 3 & 3 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find AA - BB. Also find (A-B) (A+B).

$$AA = \begin{bmatrix} 5 & 4 & 6 & 2 & 6 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 5 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and } BB = \begin{bmatrix} 4 & 1 & 0 & 3 & 5 \\ 11 & 6 & 3 & 9 & 15 \\ -4 & -2 & -1 & -3 & -5 \\ 1 & 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } AA - BB = \begin{bmatrix} 1 & 3 & 6 & -1 & 3 \\ -10 & -4 & 1 & -9 & -15 \\ 4 & 2 & 2 & 3 & 5 \\ 1 & 2 & 6 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 2 & 4 & 0 & -3 \\ -2 & -3 & -2 & -2 & -1 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and } A + B = \begin{bmatrix} 2 & 2 & 2 & 4 & 5 \\ 4 & 3 & 4 & 2 & 3 \\ -1 & -1 & -1 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{So } (A-B)(A+B) = \begin{bmatrix} 4 & 2 & 1 & 0 & 6 \\ -18 & -13 & -15 & -12 & -21 \\ 7 & 5 & 7 & 5 & 9 \\ 2 & 2 & 3 & 4 & 5 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Problem Section 2.1

1. Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 \\ -1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$

Find:

(a) AB
(b) BA
(c) AC
(d) CA
(e) BC
(f) CB
(g) A (BC)
(h) (AB) C
(i) B(C - A)
(j) BC - BA
(k) (C - A)B
(l) CB - AB
(m) AA
(n) (AA)(AA)
(o) A(AA)(AA)
(p) (AA)(AA)(AA)

(Hint: The answers to (i) and (j) will be the same and the answers to (k) and (l) will be the same, but the answers to all four need not be identical. The answer to (p) is $\begin{bmatrix} 64 & 0 \\ 2016 & 4096 \end{bmatrix}$.)

2. Perform the indicated multiplications:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(f) $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(g) $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

(h) $\begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

(i) $\begin{bmatrix} B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \\ 5 & 1 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and $\mathcal{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Compute:

(a) $A\mathcal{O}$
(b) $\mathcal{O}A$
(c) AI
(d) IA
(e) II
(f) $(II)I$
(g) IA
(h) $(AI)I$
(i) $(II)A$
(j) AIA
(k) AA
(l) $(\mathcal{O}\mathcal{O})\mathcal{O}$

We are now going to point out a surprising property of matrix multiplication which makes it very different from ordinary multiplication with numbers. If you have already discovered that multiplication of matrices is not commutative, that is, that AB does not necessarily equal BA, please have patience while we explain it to the other readers.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 6 & 9 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix}$

That is, $AB \neq BA$. In fact, it is really rather unusual when $AB = BA$. You can convince yourself of this by making up some matrices A and B and comparing AB with BA. Because matrices are not commutative (except in special cases), we will have to pay careful attention to the order in which the matrices are placed. In fact, it is best to think of multiplying on the left by a factor as "left multiplication" and multiplying on the right by a factor as "right multiplication." In the algebra of numbers "left multiplication" is the same as "right multiplication," but this is not true in the algebra of matrices; e.g., $xy = yx$ when x and y are numbers, but $CD \neq DC$ when C and D are matrices. CD may equal DC, but usually it does not.

Problem Section 2.2

1. Given $M = \begin{bmatrix} -1/2 & 3 \\ 2 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$,

find:

(a) MN
(b) NM
(c) N(MN)
(d) (MN)N
(e) (NM)M

(answer to (e)) $\begin{bmatrix} 81/2 & 109 \\ 187/2 & 239 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$,

$\bigcirc = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Find:

(a) AB
(b) BA
(c) (AB)I
(d) $\bigcirc$(BA)
(e) I(AB)

$\bigcirc$ is called the zero matrix, as was pointed out earlier, and I is called the unit matrix. In ordinary algebra, multiplication by the number 1 leaves other numbers unchanged and addition of the number 0 to other numbers leaves the other numbers unchanged. Because of this, 1 is sometimes referred to as the identity element under multiplication and 0 is called the identity element under addition. The matrices I and $\bigcirc$ behave in the very same way in matrix algebra. That is, in matrix algebra, the unit matrix and the zero matrix are identity elements under multiplication and addition, respectively. Notice also that I and $\bigcirc$ commute with every matrix under both addition and multiplication. It should be clear by now that I is always a matrix having the number 1 for elements along the principal diagonal -- i.e., the diagonal extending from upper left to lower right — and the number 0 for all of its other elements. In symbols

$[I]_{i,j} = 1$ when $i = j$ and $[I]_{i,j} = \bigcirc$ when $i \neq j$.

Problem Section 2.3

1. Given A = any 3×3 matrix, prove that $AI = IA = A$.
2. Given B = any 4×4 matrix, prove that $B\mathbf{O} = \mathbf{O}B$

We shall now indicate a second rather unusual property which matrices possess. In the ordinary algebra of numbers, $x \cdot y = 0$ if and only if, either $x = 0$ or $y = 0$. Consider, however, the matrices

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix} .$$

Their product is $\mathbf{O}$, yet neither is the zero matrix (recall that in ordinary algebra, if $ab = 0$, then either $a = 0$ or $b = 0$ or both). This strange property, along with the breakdown of the commutative law of multiplication, leads to further differences between matrix algebra and ordinary algebra. In the algebra of numbers, a real number "a" can have at most two square roots, three cube roots,..., and n n^{th} roots. Suppose, for instance, $xx = a$ and also $yy = a$. Then,

(1) $xx - yy = 0$

(2) Now $(x - y)(x + y) = xx - yx + xy - yy$

(3) But $xy = yx$

(4) So $(x - y)(x + y) = xx - yy$

(5) And $(x - y)(x + y) = 0$ (by (1))

(6) Now if $x - y = 0$, $x = y$, and if $x + y = 0$, $x = -y$

(7) So either $x = y$, or $x = -y$.

This proves that a real number "a" can have at most two square roots. For matrices, this proof would fail, for step (3) is not valid for matrices. Even if matrices were commutative under multiplication, the proof would be blocked by step (6). For matrices, then, the proof does not hold. It is not true that a matrix can have at most two square roots. The matrix I has the following four (different) square roots:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } L = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We shall check out one of these and leave the other three to the reader.

$$K\,K = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

In addition to these four the matrix $\begin{bmatrix} 0 & \lambda \\ 1/\lambda & 0 \end{bmatrix}$ (where λ is any real number except 0) is also a square root of I.

Proof:

$$\begin{bmatrix} 0 & \lambda \\ 1/\lambda & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ 1/\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Since there are infinitely many numbers in the real number system, I has infinitely many square roots. In this respect, the unit matrix of matrix algebra is quite different from the number one in the real number system, for the latter has but two square roots; viz., 1 and -1. This illustration should make it clear that the fact that a number has at most n n^{th} roots is a useful and important consideration.

Problem Section 2.4

1. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Show that $AB \neq BA$ and thereby show that :

(a) $(A + B)\,(A + B) \neq AA + 2AB + BB$

(b) $(A - B)\,(A + B) \neq AA - BB$

(c) $\left[(A + B)\,(A + B)\right]\,(A + B) \neq (AA)A + 3(AA)B + 3\,A(BB) + (BB)\,B$

2. Find at least four square roots of the matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How many more do you suspect there are?

3. Show that $A = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ (n = any real number)

satisfies the equation $AA = O$.

How many different 2 x 2 matrices like A satisfy the equation $AA = O$? List three of them (with numbers rather than letters for elements).

4. If you feel that you understand the effect of multiplying by I, you need not bother with the following.

Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$, find:

(a) AI
(b) A (I I)
(c) (IA) (IA)
(d) AA

One way to prove that CI = C, where C = any 3 x 3 matrix, is to:

$$\text{Let } C = \begin{bmatrix} X_1 & X_2 & X_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\text{Then } CI = \begin{bmatrix} X_1 & X_2 & X_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & X_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = C$$

Another way is to use the notation for the elements of a matrix. This, in fact, is perhaps a better way, for the proof can be given when C is an n x n matric just as easily as when C is a 3 x 3 matrix. At the risk of boring the reader, let us once again say that by I_n we mean an n x n matrix which has the number 1 as elements along the principal diagonal and the number 0 elsewhere.

Example:

$$I_n = \begin{array}{cl} \text{n of them} & \\ \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} & \begin{array}{l} \text{n} \\ \text{of} \\ \text{th} \\ \text{em} \end{array} \end{array}$$

As we did earlier with the zero matrix, we shall again abandon the unnecessary subscript n, for the size of I is always evident from context.

Theorem 2.1: Let A be an n x n matrix. Then AI = IA = A

Proof: Using our definition for multiplication, the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of AI is

$$[AI]_{i,j} = [A]_{i,1} \cdot [I]_{1,j} + [A]_{i,2} \cdot [I]_{2,j} + \dots + [A]_{i,n} [I]_{n,j}.$$

From the definition of the unit matrix, $[I]_{i,j} = 0$ unless $i = j$; therefore, all of the terms on the right-hand side of the above equation will vanish except the one containing $[I]_{i,j}$ with $i = j$. Thus, we have

$$[AI]_{i,j} = [A]_{i,j} [I]_{j,j}.$$

But $[I]_{j,j} = 1$, so the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of AI must be $[A]_{i,j}$. But this is identical with the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of A. Since this is true for all i and j from 1 to n, every element in AI is identical to the corresponding element of A. It follows that AI = A. In a similar way it can be shown that IA = A. Then, since both AI and IA are equal to A, AI and IA are equal to one another.

Problem Section 2.5

1. Prove that $IA = A$ for all A.

Two of the basic laws which govern multiplication of ordinary numbers have failed for matrices. The first is the commutative law of multiplication and the second is the law of cancellation. The breakdown of the latter can be written as follows:

$$AB = O \text{ and yet } A \neq O \text{ and } B \neq O \quad .$$

The failure of these two important laws might lead the student to anticipate the crumbling of the entire structure which we are trying to erect, but such is not the case. With the exception of these two laws, the basic laws of number algebra have an analogous counterpart in matrix algebra. The other basic laws are listed in

Theorem 2.2: If A, B, and C are nxn matrices, then:

(1) $(AB)C = A(BC)$ (associative law of multiplication)

(2) $A(B + C) = AB + AC$ (left distributive law*)

(3) $(B + C)A = BA + CA$ (right distributive law)

We shall prove (2) and leave it to the reader to prove (3). Note that if the commutative law of multiplication were valid for matrices, parts (2) and (3) of the theorem would be identical statements, and there would be no need to prove the two parts separately.

One can prove (1) in a way similar to the proof we shall give for (2) (i.e., by expanding and rearranging), but since the details are rather messy, we shall omit it.

Proof of (2): Using matrix element notation, the element in the i^{th} row, j^{th} column of $A(B + C)$ is

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* By distributive law, we mean "the distributive law of multiplication with respect to addition." In some systems there is also a "distributive law of addition with respect to multiplication," but no such law holds for matrices.

$$\text{(i)}\ [A(B+C)]_{i,j} = [A]_{i,1}\,[B+C]_{1,j} + [A]_{i,2}\,[B+C]_{2,j} + \cdots + [A]_{i,n}\,[B+C]_{n,j}\,.$$

$$\text{(ii)}\ [A\,(B+C)]_{i,j} = [A]_{i,1}\left\{[B]_{1,j} + [C]_{1,j}\right\} + [A]_{i,2}\left\{[B]_{2,j} + [C]_{2,j}\right\} + \cdots + [A]_{i,n}\left\{[B]_{n,j} + [C]_{n,j}\right\}$$

$$\text{(iii)}\ = \left\{[A]_{i,1}\,[B]_{1,j} + \cdots + [A]_{i,n}\,[B]_{n,j}\right\} + \left\{[A]_{i,1}\,[C]_{1,j} + \cdots + [A]_{i,n}\,[B]_{n,j}\right\}$$

$$\text{(iv)}\ = [AB]_{i,j} + [AC]_{i,j}$$

$$\text{(v)}\ = [AB + AC]_{i,j}$$

In step (i) we used the definition of the product of two matrices. Step (ii) is based upon the definition of the sum of two matrices. The third step is actually two small steps. First, the $[A]$ preceding each pair of braces was multiplied by the two terms within the braces (distributive law) and then all terms containing the letters A and B were placed within one set of braces and all terms containing the letters A and C were arranged in another set of braces. The reader should realize at this point that the first "braced" expression in step (iii) is actually the element in the i^{th} row and j^{th} column of the matrix AB; that is, it is $[AB]_{i,j}$. In a similar way, the expression in the second set of braces is $[AC]_{i,j}$. This gives step (iv) and applying the definition of matrix addition to (iv) gives us the fifth and final step. Since we have proved that the left distributive law holds for the general element—i.e., the element in the i^{th} row and j^{th} column — we have proved that it holds for all elements; therfore, A(B + C) = AB + AC.

By part (1) of theorem 2.2, A (BC), (AB) C, and ABC are one and the same matrix. Parts (2) and (3) justify our "multiplying out," but care must be taken to preserve the correct order of the factors.

The reader should always pay close attention to the order in which the matrices are written. For example, $ABAB$ may not equal $(AA)(BB)$ and it is more likely than not that $AB + AC \neq BA + CA \neq BA + AC \neq AB + CA$.

We now compare matrix algebra and ordinary algebra of numbers in the following ways:

(1) They are very much the same for addition and subtraction.

(2) The procedure of "multiplying out" with ordinary numbers can also be used for matrices, but only if the correct order of the factors is preserved. The factors in a product cannot be rearranged unless it is explicitly stated that the matrices are commutative.

(3) AB can equal the zero matrix, even if both $A \neq O$ and $B \neq O$, whereas for ordinary algebra $ab = o$ implies that either $a = o$ or $b = o$ (or both).

Problem Section 2.6

1. Prove that $(B + C)A = BA + CA$, where A, B, and C are n x n matrices.

2. Given $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$,

test the validity of the following formulae:

(a) $A(B + C) = BA + AC$

(b) $(B + C)A = BA + CA$

(c) $A(B + C) = AB + AC$

(d) $(B + C)A = AB + CA$

3. (a) Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Show that $AB \neq O$ but $BA = O$.

(This illustrates both the failure of the commutative law of multiplication and the breakdown of the law of cancellation.)

(b) Devise two 2 x 2 matrices which illustrate the same principles as part (a) .

4. Given

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix},$$

(a) Show that AB = BA for all a, b, c, d.

(b) Illustrate this by assigning arbitrary values to a, b, c, and d.

5. Given

$$W = \begin{bmatrix} wx & x^2 \\ -w^2 & -wx \end{bmatrix},$$

(a) Show that $WW = 0$.

(b) Assign numbers to w and x and thereby illustrate the validity of $WW = 0$ for these numbers.

6. If GH = -HG, G and H are said to be anticommutative.

(a) Test the following matrices for anticommutativity.

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$$

(Six products must be computed here)

(b) Illustrate the pairs which are anticommutative by assigning a numerical value to "a ."

7. Show that the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

satisfies the equation $AA - 5A + 2I = 0$

8. Show that the matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

satisfies the equation $(BB)B - 2(BB) + B = 0$.

9. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix},$$

and $C = \begin{bmatrix} 0 & 0 & 0 \\ 2a & 0 & 0 \\ a & 2a & 0 \end{bmatrix}$. Show that $AB = BA$, $AC = CA$, and $BC = CB$.

Now let $D = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$.

Show that D does not commute with the above three matrices; i.e., show that $AD \neq DA$, $BD \neq DB$, and $CD \neq DC$.

Since matrix multiplication and ordinary multiplication are similar in many important ways, we can define the powers of a matrix in the ordinary way. Our definition shall lead us to the basic law of exponents.

Definition 2.2: We define A^2 as AA, A^3 as AA^2, A^4 as AA^3, etc. More concisely, we make the following inductive definition: $A^2 = AA$, $A^{n+1} = AA^n$. We also define A^1 as A and A^0 as I.

Theorem 2.3: $A^nA^m = A^{n+m}$, where A = any (square) matrix and n and m are integers.

Proof: We shall prove the theorem by mathematical induction (on n). For $n = 1$, we have $A^1A^m = A^{1+m}$

or $AA^m = A^{m+1}$, and this is a true statement.

Now assume the theorem is true for $n = k$.

Then $A^kA^m = A^{k+m}$

We now use this assumption to show that the theorem is true for $n = k + 1$.

$$A^{k+1} = AA^k \text{ (by definition)}$$

$$\therefore A^{k+1}A^m = (AA^k)A^m \text{ (multiplied through by } A^m\text{)}$$

$$A^{k+1}A^m = A(A^kA^m) \text{ (by the associative law)}$$

$$A^{k+1}A^m = A(A^{k+m}) \text{ (by inductive assumption)}$$

$$A^{k+1}A^m = A^{(k+m)+1} \text{ (by definition)}$$

$$A^{k+1}A^m = A^{(k+1)+m} \text{ (by associative and commutative laws for ordinary algebra)}$$

(Now we know the theorem is true when $n = 1$, and we also know that if the theorem is true for $n = k$, then it is true for $n = k + 1$. So, if $k = 1$, then it is true for $n = 2$; then, let $k = 2$, and it follows that it is true for $n = 3$. This process does not terminate; therefore, the theorem holds for all n).

The theorem follows by induction.

It should be noted that this basic law of exponents applies only to powers of a single matrix A. For example, A^2BA does not necessarily equal A^3B. The factors in such a product must not be rearranged. If, however, we have $A^2AA^3B^4B^2A^3A^4$, we can rewrite it as $A^6B^6A^7$, for this involves only the law of exponents and the order of the factors has not been changed.

Problem Section 2.7

1. In matrix algebra $(A + B)(A - B) \neq A^2 - B^2$ except in special cases. Why is this so? Devise two matrices which will illustrate the above inequality. (Note: Just one example is necessary to disprove the proposition $(A + B)(A - B) = A^2 - B^2$. Such an example is called a counter-example). Devise two other matrices which will illustrate an exception to the above inequality. (You may use matrices of size 2).

2. Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}.$$

Find the following matrices:

(a) AAB (d) BBA
(b) ABA (e) BAB
(c) BAA (f) ABB

Which of these are the same? Now suppose that A and B were numbers rather than matrices. Which expressions would then be the same?

3. A matrix A is called nilpotent if some power of A is 0. Show that

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \text{ is nilpotent.}$$

4. If $A^2 = A$, A is said to be an idempotent matrix. Show that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is idempotent.}$$

5. With the exception of the zero matrix, an idempotent matrix is never nilpotent. Explain why this is so.
(Hint: If $A^2 = A$, then $A^n = A$)

6. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Compute A^5 in three different ways. (e.g. — AAAAA, A^2A^2A, etc.)

7. Given

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ compute } M^2, M^3, \text{ and } M^4.$$

We have proposed two different kinds of multiplication in this program. In Chapter 1, the definition of the multiplication of a number times a matrix was given and in the present chapter we have defined the multiplication of a matrix by a matrix. Both types of multiplication are involved in

Theorem 2.4 : If A and B are matrices of the same size and if x is a real number, then $x(AB) = (xA)B = A(xB)$.

This is an important theorem for it tells us that we need not worry about the position of numerical factors in a matrix product.

Example:

$$(6A)\ (3A)\ (B)\ (4) = 72\ A^2\ B.$$

Note, however, that we continue to pay careful attention to the order in which the matrices come.

Now, using several of the properties of matrices which have thus far been developed, we shall show that the binomial theorem does not hold for (non-commutative) matrices. As was pointed out earlier, only one counter-example is needed to "disprove" a mathematical proposition. The binomial theorem for ordinary algebra is:

$$(a + b)^n = a^n + na^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \dots + b^n.$$

In the algebra of numbers, this theorem is useful for expanding expressions like $(a + b)^4$. Of course one can always "multiply out" the expression; i.e., $(a + b)\ (a + b)\ (a + b)\ (a + b)$. But when we are confronted with an expression like $(x + y/2)^9$, "multiplying out" becomes rather tedious. Using the binomial theorem we just "write down" the expansion.

Example: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. "Multiplying out," we have $(a+b)^3 = (a+b)(a+b)(a+b) = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$.

Now consider the analogous problem with matrices.

$$(A+B)^3 = (A+B)\ (A+B)\ (A+B) = (A^2 + AB + BA + B^2)\ (A+B)$$
$$= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3.$$

Since matrices are not commutative, this is as far as we can go, and this simple counter-example rules out the possibility of there being a binomial theorem for (non-commutative) matrices.

We shall now backtrack for a moment in order to give a proof of theorem 2.4. We again make use of our notation for the elements of a matrix to do so.

$$\begin{aligned}
[(xA)B]_{i,j} &= [xA]_{i,1}\,[B]_{1,j} + [xA]_{i,2}\,[B]_{2,j} + \\
&\quad \ldots + [xA]_{i,n}\,[B]_{n,j}\,. \\
&= \left\{x[A]_{i,1}\,[B]_{1,j}\right\} + \left\{x[A]_{i,2}\,[B]_{2,j}\right\} \\
&\quad + \ldots + \left\{x\,[A]_{i,n}\,[B]_{n,j}\right\}. \\
&= x\left\{[A]_{i,1}\,[B]_{1,j} + [A]_{i,2}\,[B]_{2,j} + \right. \\
&\quad \left. + \ldots\,[A]_{i,n}\,[B]_{n,j}\right\}. \\
&= x\,[AB]_{i,j} \\
&= [x(AB)]_{i,j}
\end{aligned}$$

We have proved this by making use of the product definition, applying definition 1.4, applying (4) of theorem 1.4, applying the product definition again, and, finally, using definition 1.4. Since we have proved it is true for the element in the $i^{\underline{th}}$ row, $j^{\underline{th}}$ column, it is true for all of the elements, thus $(xA)B = x(AB)$.

Problem Section 2.8

1. We have shown that the binomial theorem does not generally hold for matrices by using a counter-example. Think of another counter-example which will show the same thing.

2. The binomial theorem holds in certain special cases.

   (a) Expand $(A + 3I)^4$ by "multiplying out."

   (b) Expand $(A + 3I)^4$ by the binomial theorem.

(c) Why does the binomial theorem "work" in part (b) ?

(d) Prove that the binomial theorem can be used to expand

$(A + bI)^n$.

3. Multiply out:

(a) $(A + 3B)^2(A - 2B)$

(b) $(A - B + C)^2 \, (A - C)^2$

(c) $(A + 1/3\ B)^4$

4. Show that

$$D = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{satisfies} \quad D^2 - 3D + 2I = O$$

5. Show that

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

satisfies the equation $W^3 = 2W^2 - I$

6. Show that the matrix

$$B = \begin{bmatrix} 1 & 2x \\ \frac{2}{x} & 1 \end{bmatrix} \quad \text{satisfies the}$$

equation $B^2 - 2B - 3I = O$ for all real numbers x except $x = 0$ and thereby demonstrate that this quadratic equation has infinitely many solutions (roots).

7. Prove that $(xA)B = A(xB)$

(Hint : This proof is quite similar to the one we have given for $(xA)B = x(AB)$.

Since:

(1) The law of exponents is the same for matrices as it is for numbers,

(2) Numerical factors in a matrix product can be rearranged, and

(3) The laws governing addition and subtraction of matrices are identical with those governing addition and subtraction of numbers,

matrix polynomials which are composed of sums and products of a single matrix A can be manipulated just as ordinary polynomials in a single letter x.

Example:

$$(A + 2I)^3 = A^3 + 3A^2 (2I) + 3A (2I)^2 + (2I)^3$$
$$= A^3 + 6A^2 + 12A + 8I$$

just as

$$(x + 2)^3 = x^3 + 3x^2 (2) + 3x (2)^2 + (2)^3$$
$$= x^3 + 6x^2 + 12x + 8 .$$

also,

$$(A - I)^2 (A + I) = (A - I) (A^2 - I)$$
$$= A^3 - A^2 - A + I$$

just as

$$(x - 1)^2 (x + 1) = (x - 1) (x^2 - 1)$$
$$= x^3 - x^2 - x + 1 .$$

The general form of such polynomials in A can be written as $a_m A^m + a_{m-1} A^{m-1} + \dots + a_0 I$, where $a_m, a_{m-1}, \dots a_0$ are numerical coefficients. If the a_m, am-1, ...,a_0 were matrices rather than numbers, the non-commutative nature of matrices would complicate things, but we shall confine ourselves to numerical coefficients throughout this program. If the general polynomial is set equal to $\bigcirc$, i.e., if $a_m A^m + a_{m-1} A^{m-1} + \dots + a_0 I = \bigcirc$ we have a polynomial equation satisfied by the matrix A. Many important properties of a matrix can be discovered by studying such equations, as we shall see in the next chapter.

Problem Section 2.9

1. Prove that $A^{n} - I = (A - I)(A^{n-1} + A^{n-2} + \dots + I)$

2. Factor the following polynomials:

(a) $A^2 - I$ (d) $A^3 - A^2 - 2A$

(b) $A^3 + 3A^2 + 3A + I$ (e) $A^4 - 5A^2 + 4I$

(c) $A^2 - 4A + 4I$

3. You now have each of the matrix polynomials in problem # 2 (above) as a product of first degree polynomials. Set each product equal to O, thus making them polynomial equations, and find their roots.

(Answers: (d) is satisfied by $A = O$, $A = 2I$, and $A = -I$.)

CHAPTER 3

Division of Matrices

In the first two chapters of this program we have studied addition, subtraction, and multiplication of matrices. We shall now consider division of matrices. In ordinary algebra, division is necessary to solve equations like $ax = b$. There is a similar problem with matrices, viz., solving equations like $AX = B$. Since the commutative law of multiplication breaks down for matrices, AX need not equal XA; therefore, $AX = B$ and $X'A = B$ are different problems. That is, if we solve both of these equations, it is not likely that X will equal X'.

Example:

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Then, $AX = B$ becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$ and has the solution

$$X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix},$$

whereas $X'A = B$ becomes

$$X' \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$ and has the solution

$$X' = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

(Substitute X and X' back into their respective equations and thereby show that they satisfy them.)

The fact that we must distinguish between right-hand and left-hand division might have been anticipated, since we had a right-hand and left-hand product when we studied multiplication. An even more fundamental difficulty than this is in store for us, however.

Consider, for example, the following equation.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

Now let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$

No matter what numerical values we take for the elements of X, the second row of the product

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must be all zeros.

Therefore, the product cannot possibly equal $\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$.

Thus the proposed equation has no solution. This is similar to saying that division by $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is impossible, just as division by 0 is impossible in ordinary algebra. How, then, can we hope to have any kind of "systematic" division of matrices? Ordinary algebra gives us a cue. If, for a matrix A, there exists a matrix A^{-1} such that $A^{-1}A = I$, then we can solve the equation $AX = B$ as follows:

$$AX = B$$

$$A^{-1}AX = A^{-1}B \text{ (multiplied through on the left by } A^{-1}\text{)}$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Thus $X = A^{-1}B$ is the solution to the equation $AX = B$. Since A^{-1} behaves very much like a reciprocal in ordinary algebra, we shall call it the reciprocal matrix of A. From all of this the reader might surmise that A^{-1} exists if and only if $AX = B$ has a solution. This, it turns out, is precisely the case.

Example:

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Since $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I$,

$$A^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

Making use of this information we can solve an equation like $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$. Multiplying on the left by A^{-1} gives

$$\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix},$$

so $X = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$.

The alert reader may have noticed that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I,$$

and so $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$

becomes $3IX = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$

or $3X = \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$

or $X = 1/3 \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$.

In our next illustrative example, however, no such "trick" is possible.

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$.

Then, since $\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} = I$,

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}.$$

Suppose $\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$.

Multiplying on the left by A^{-1} gives

$$\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} X = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$$

or $X = \begin{bmatrix} 2 & 11 \\ -1 & -9 \end{bmatrix}$.

From what has been said it should be apparent that the matrix we considered earlier, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, has no reciprocal, for if it had a reciprocal, we could have solved the equation

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}.$$

We have used the word "reciprocal" to stand for A^{-1} since this word carries meaning which makes it easier for the reader to recognize the similarity between division of matrices and ordinary division. Most books, however, prefer to call A^{-1} the inverse of A. We shall use the words "inverse" and "reciprocal" interchangeably in this program.

The problem of division of matrices is actually the problem of finding the reciprocal or inverse of a given matrix. The remainder of Chapter 3 is devoted to : (1) Finding out whether or not a given matrix has an inverse, and (2) discovering a way to calculate inverses (provided the inverses exist).

Problem Section 3.1

1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Show that $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ provided that $ad - bc \neq 0$.

Notice that $ad - bc = |A|$ (" $|A|$ " read "the determinant of A"). The above formulation tells us that a 2 x 2 matrix has an inverse if and only if its determinant does not equal zero. This turns out to be true for all (square) matrices.

2. Use the matrices A and A^{-1} in #1 (above) to find the reciprocals (inverses) of the following matrices. One of the matrices has no inverse. Which one is it?

(a) $A = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}$ (c) $C = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ (d) $D = \begin{bmatrix} 1/2 & 2/3 \\ 1/5 & 0 \end{bmatrix}$

3. Use the information in the previous problem to solve the following matrix equations for Y.

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} Y = \begin{bmatrix} 3 & 1 \\ 0 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} Y = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1/2 & 2/3 \\ 1/5 & 0 \end{bmatrix} Y = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

4. Show that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is its own inverse.

That is, show that $A = A^{-1}$.

5. Use problem #4 to solve

$$AX = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

6. We shall see a bit later that if $A^{-1}A = I$, then $AA^{-1} = I$. In words: we shall see that every matrix commutes with its inverse. Use this fact, together with problem #4, to solve

$$YA = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Notice that $Y \neq X$ provides further evidence for the fact that right-hand and left-hand division must be distinguished.

7. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $B = A^{-1}$.

8. Use the information given in the above problem to solve the following equations.

(a) $ZA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ (b) $AW = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$

(Notice that $Z \neq W$)

9. Prove that if a matrix A is nilpotent it can have no inverse. (Hint: first consider $A^2 = AA = O$, then generalize to $A^n = O$.

10. Make use of the idea presented in problem #9 to demonstrate that

$$B = \begin{bmatrix} 0 & d & b & a \\ 0 & 0 & e & c \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ has no inverse.}$$

Consider a 2 x 2 matrix like,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

Here,

$$A^2 = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}.$$

If I is subtracted from the first equation, the result is

$A - I = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$. Subtracting 7I from the second equation gives $A^2 - 7I = \begin{bmatrix} 0 & 12 \\ 18 & 24 \end{bmatrix}$. The I's have been subtracted in order to make the element in the upper left hand corner of both A and A^2 equal to zero.

We shall next attempt to get a matrix with zeros in the entire first row. This is accomplished by subtracting 6(A - I) from $A^2 - 7I$. Such an operation results in a matrix which has zeros not just in the first row, but everywhere. That is,

$$(A^2 - 7I) - 6(A - I) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Simplifying, we get the following polynomial equation in A.

$$A^2 - 6A - I = O$$

Once this equation has been obtained from A, we can use it to find out other things about A.

$$\begin{aligned} \text{Since } A^2 - 6A - I &= O, \\ A^2 - 6A &= I, \\ \text{and } A(A - 6I) &= I \end{aligned}$$

Then A - 6I must be the inverse of A. Let us now check this out.

$$A - 6I = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

If $A - 6I = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ is A^{-1},

then $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ should equal I. It is apparent that it does!

Thus, once a polynomial equation satisfied by a matrix is found, one can easily calculate the inverse of the matrix. The polynomial equation of a matrix can also be used to gain other information about the matrix.

Example :

Suppose $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and we wish to obtain A^8. One can always use the pedestrian approach; viz., $A^8 = AAAAAAAA$. But this is tedious and unnecessary. Such a solution requires 64 multiplications and 32 additions. A more expedient method is:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A - I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\text{and } A^2 - I = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\begin{aligned} A^2 - I - 2(A - I) &= O \\ A^2 - 2A + I &= O \end{aligned}$$

(at this point we could calculate A^{-1} if we so desired)

$$\text{Now } A^2 = 2A - I$$

$$\begin{aligned}
\text{So } A^4 &= (2A - I)^2 \\
&= 4A^2 - 4A + I \\
&= 4(2A - I) - 4A + I \\
&= 4A - 5I
\end{aligned}$$

and, finally,

$$\begin{aligned}
A^8 &= (4A - 5I)^2 \\
&= 16A^2 - 40\,A + 25I \\
&= 16\,(2A - I) - 40A + 25\,I \\
&= -8A + 9I \\
&= -8 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}
\end{aligned}$$

$$A^8 = \begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix} .$$

The reader may wish to work this problem out by repeated multiplication and compare the amount of work done to that of the above solution. There is also greater probability of arithmetical error when repeated multiplication is used.

We reiterate: once the polynomial equation of a matrix is obtained, other properties of the matrix, can be studied more handily.

The following theorem is central to what we have been discussing and is also of great importance to our future work.

Theorem 3.1 : If A is an $n \times n$ matrix, then A satisfies a polynomial equation of the form $A^k + a_{k-1}A^{k-1} + \dots + a_o I = O$ of degree $k \leq n$ and with numerical coefficients $a_{k-1}, a_{k-2}, \dots, a_o$.

Since the degree of the polynomial can never be greater than the size of the matrix which satisfies it, a 2 x 2 matrix always satisfies either a first degree (linear) or a second degree (quadratic) equation; a 3 x 3 matrix always satisfies either a linear or a quadratic or a third degree (cubic) equation; an n x n matrix, then, must satisfy some polynomial of degree k, where $1 \leq k \leq n$.

We must ask the reader to accept this theorem, for its proof is beyond the scope of this program.

Problem Section 3.2

1. Find a quadratic equation satisfied by the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}.$$

Use the equation to find A^{-1}.

2. Find a quadratic equation satisfied by $A = \begin{bmatrix} 2 & 1 \\ 5 & -8 \end{bmatrix}$

Use the equation to find A^{-1}.

Use the equation to find A^6.

3. Find a quadratic equation which has

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ as a root.}$$

Find B^{-1}.

Find B^{16}. (Answer: $B^{16} = \begin{bmatrix} 1597 & 987 \\ 987 & 610 \end{bmatrix}$)

4. Given $M = \begin{bmatrix} 1/3 & 7 \\ 1/7 & 3 \end{bmatrix}$, find a quadratic equation satisfied by M.

Does M^{-1} exist?

5. Show that the matrix $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ satisfies the equation

$A^2 = -4I$. Use this information to obtain B^{4000}, where $B = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$.

(answer: $B^{4000} = 2^{600} I = \begin{bmatrix} 2^{600} & 0 \\ 0 & 2^{600} \end{bmatrix}$)

Theorem 3.1 assures us that every n x n matrix is a root of a polynomial equation of degree at most n. This, however, does not preclude the possibility that a matrix may satisfy more than one such equation.

Example:

Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then A satisfies $A^2 = O$, $A^3 = O$, $A^4 = O$, and $A^5 = O$.

It should be clear that the equation of lowest degree which a matrix satisfies is the equation which gives the most information about the matrix. Therefore, we shall always be seeking the equation of smallest possible degree which a matrix satisfies. We shall subsequently prove that the equation of smallest possible degree which a matrix satisfies is unique; that is, there is one and only one such equation of least degree for each matrix. This idea is important to our work, so we make the following definition.

Definition 3.1: The polynomial of smallest possible degree which a matrix A satisfies is called the least equation satisfied by A.

We shall now give a systematic procedure for finding the least equation of a matrix. Consider first matrices of size 2. Given a matrix A, A^2 is easily computed. Having A, A^2, and I, one can always find the least equation satisfied by a 2 x 2 matrix A as follows: Find a number "a" such that A - aI is a matrix with element 0 in the upper left-hand corner. If it happens that A - aI is the zero matrix, we are done, for in this case A satisfies the linear polynomial equation $A - aI = O$. If, however, any element of $A - aI$ is different from 0, we continue the process by finding a number "b" such that $A^2 - bI$ is a matrix having 0 in the upper left-hand corner. We now have two matrices, which have 0 in the upper left-hand corner; viz., $A - aI$ and $A^2 - bI$. Now, for 2 x 2 matrices, if $A - aI \neq O$, then there exists a number "c" such that

$$(A^2 - bI) - c(A - aI) = O .$$

This latter equation is the one we are seeking.

Example : Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

Then $A^2 = \begin{bmatrix} 20 & 40 \\ 40 & 80 \end{bmatrix}$. The number corresponding to "a" in the above description is 2

for $A - 2I = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}$. Since $A - 2I$ has at least one element different from 0, $A - 2I$ is not the polynomial we seek, so we are obliged to continue. The number 20 corresponds to "b" for

$$A^2 - 20I = \begin{bmatrix} 0 & 40 \\ 40 & 60 \end{bmatrix} .$$

By inspection, each element of $A^2 - 20I$ is exactly 10 times each corresponding element of $A - 2I$. So $c = 10$ and the least equation of A is

$$A^2 - 20I - 10(A - 2I) = O$$

or $A^2 - 10A = O$ (Notice that A has no reciprocal.)

The value of A can always be substituted into the equation to see if A does satisfy it.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} - 10 \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \overset{?}{=} O$$

$$\begin{bmatrix} 20 & 40 \\ 40 & 80 \end{bmatrix} - \begin{bmatrix} 20 & 40 \\ 40 & 80 \end{bmatrix} = O$$

There is no intrinsic reason for our making the upper left-hand element equal to 0. One could just as well make some other element zero. In this program, however, in order to be consistent, we shall always start in the upper left-hand corner and work from left to right. The next example, which will consider the same problem with a 3 x 3 matrix, should help to clarify what we are saying here.

Example:

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Now $A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $A^2 - 2I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$,

and $A^3 - 4I = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$.

We now have three matrices with the number 0 in the upper left-hand corner. We wish to subtract multiples of $A - I$ from each of the latter two matrices so that the two resulting matrices have 0's in row one column two as well as in row one column one.

$$(A^3 - 4I) - 2(A - I) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $(A^2 - 2I) - (A - I) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ do the "trick ."

Those two matrices happen to be identical so their difference is the zero matrix.

$$(A^3 - 4I) - 2(A - I) - (A^2 - 2I) + (A - I) = \bigcirc \qquad \text{or}$$

$A^3 - A^2 - A - I = \bigcirc$, which is the least equation satisfied by A.

To find A^{-1} ,

$$A(A^2 - A - I) = I$$

so $A^{-1} = A^2 - A - I$

or $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

When we have more than one equation satisfied by a particular matrix, we simply select the one of least degree.

Example :

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Also, $A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 - I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $A^3 - I = \begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Subtracting suitable multiples of $A - I$ from the latter two matrices gives

$$(A^2 - I) - 2(A - I) = O$$

$$(A^3 - I) - 3(A - I) = O.$$

Now both of these are polynomial equations satisfied by A but the least equation is

$$(A^2 - I) - 2(A - I) = O$$

$$\text{or} \quad A^2 - 2A + I = O.$$

The preceding procedure, being general, can be applied to matrices of any size. Let A be an n x n matrix. We first calculate $A^2, A^3, \ldots, A^n$. We then find numbers $a_1, a_2, \ldots, a_n$ such that the matrices $B_1 = A - a_1 I$, $B_2 = A^2 - a_2 I, \ldots, B_n = A^n - a_n I$ all have the number 0 in the upper left-hand corner. If $B_1 = O$, we are done, for then $A - a_1 I = O$ is the least equation we are seeking. If not, we find a second set of numbers $b_1, b_2, \ldots, b_n$ such that the matrices $C_2 = B_2 - b_1 B_1, \ldots C_n = B_n - b_n B_1$ have zeros for elements $[C]_{1,2}$ as well as for $[C]_{1,1}$. If $C_2 = O$ — i.e., if $(A^2 - a_2 I) - b_2(A - a_1 I) = O$ —, we have the least equation satisfied by A; if not the process is continued. It can be proved that this process terminates after (at most) k steps, where $k \leq n$, and if W_y is the first of the matrices B_1, C_2, D_3, etc., which is equal to O, then $W_y = O$ is the least equation in A. The proof is not too difficult in principle but is rather messy, so we shall omit it.

Problem Section 3.3

1. Given $A = \begin{bmatrix} \sqrt{2} & 3 \\ -4 & 0 \end{bmatrix}$, find the least equation satisfied by A. Find A^{-1}.

2. Find the least equation satisfied by

$$z = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{12} \end{bmatrix}. \text{ Find } z^{-1}. \text{ Find } z^4.$$

(Hint: $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$)

(Answer: $z^4 = \begin{bmatrix} 9 & 0 \\ 0 & 144 \end{bmatrix}$)

3. Find the least equation satisfied by $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Find M^{-1}.

Now find the least equation satisfied by M^{-1}. Use this latter equation to find the inverse of M^{-1}; that is, use it to find $(M^{-1})^{-1}$. What is the inverse of the inverse of M?

4. Find the least equation satisfied by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}.$$

Use the least equation to find A^{-1} and A^8.

5. Find the least equation satisfied by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

What is A^{-1}? (This one should take you less than one minute.)

6. Given

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ find } C^{-1}.$$

Check the C^{-1} you obtain by showing that $C^{-1}C = I$

and $CC^{-1} = I$.

The reader now knows how to find (at least in principle) the least equation of any square matrix. Perhaps by now it has become apparent that one can tell whether or not a matrix has an inverse by briefly glancing at the least equation satisfied by the matrix. If not, we clarify this matter with

Theorem 3.2 : Let $A^k + a_{k-1}A^{k-1} + \dots + a_0 I = \bigcirc$ be the least equation satisfied by A. Then A has an inverse if and only if $a_0 \neq 0$. In fact, if $a_0 \neq 0$, $A^{-1} = -\frac{1}{a_0}\ (A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I)$.

The theorem is proved in two parts. In part (1) $a_0 \neq 0$ is assumed, and it is proved that A^{-1} exists. Part (2): $a_0 = 0$ is the assumption and it is proved that A^{-1} does not exist.

Part (1) : Let $A^k + a_{k-1}A^{k-1} + \dots + a_0 I = \bigcirc$ be the least equation satisfied by A and let it be stipulated that $a_0 \neq 0$. Then A has an inverse.

Proof: Adding $-a_0 I$ to both sides of the given polynomial equation gives

$$A^k + a_{k-1}A^{k-1} + \dots + a_1 A = -a_0 I .$$

Factoring out an A on the left-hand side of the equation, we have

$$(A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I)A = -a_0 I .$$

Dividing both sides of the equation by $-a_0$ (this can be done since it was specified that $a_0 \neq 0$) results in

$$\left[-\frac{1}{a_0}(A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I) \right] A = I$$

This makes it clear that A does have an inverse, viz., the expression within the brackets.

Part (2): Let $A^k + a_{k-1}A^{k-1} + \dots + a_0 I = \bigcirc$ be the least equation satisfied by A and let it be stipulated that $a_0 = 0$. Then A^{-1} does not exist.

Proof : Since $a_0 = 0$ we had best rewrite the given equation as

$$A^k + a_{k-1}A^{k-1} + \dots + a_1 A = O \; .$$

We again factor out an A, this time on the left, and get

$$A(A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I) = O \, .$$

Now assume that A has an inverse A^{-1}. Multiplying both sides of the equation (on the left) by A^{-1} gives

$$(A^{-1}A)(A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I) = O$$

or $(I)(A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I) = O$

or just $A^{k-1} + a_{k-1}A^{k-2} + \dots + a_1 I = O$.

But now we have an equation of degree k-1 satisfied by A which contradicts the fact that the least equation satisfied by A was of degree k. Thus assuming that A^{-1} exists when $a_0 = 0$ leads to a contradiction. We conclude that A^{-1} does not exist when $a_0 = 0$. This completes the proof of theorem 3.2.

Since the commutative law of multiplication does not hold for matrix algebra, the reader might fear that it is necessary to distinguish between right and left inverses. This matter is attended to in

Theorem 3.3 : If A is any (square) matrix having an inverse A^{-1}, and if $A^{-1}A = I$, then AA^{-1} also equals I. That is, if $A^{-1}A = I$, then $A^{-1}A = AA^{-1} = I$.

Proof: It is clear that

$$A^{-1}(AA^{-1}) = (A^{-1}A)(A^{-1}) = IA^{-1} = A^{-1}.$$

Therefore

$$(A^{-1})^{-1}\left[A^{-1}(AA^{-1})\right] = (A^{-1})^{-1}(A^{-1}) = I.$$

$$\text{Also } (A^{-1})^{-1}\left[A^{-1}(AA^{-1})\right] = \left[(A^{-1})^{-1}(A^{-1})\right](AA^{-1})$$

$$= I(AA^{-1})$$

$$= AA^{-1}$$

Thus $AA^{-1} = I$ and so $A^{-1}A = AA^{-1} = I$.

This reveals that every matrix commutes with its inverse so the equations $XA = I$ and $AY = I$ have the same solution ; viz.,

$$A^{-1} = X = Y .$$

We know from theorem 3.1 and definition 3.1 that every matrix satisfies at least one least equation. We now shall show that every matrix satisfies at most one least equation.

Theorem 3.4 : The least equation satisfied by a matrix is unique.

Proof : Suppose two such least equations of degree k are satisfied by a matrix A but no equation of degree j where $j < k$ is satisfied by A. Let the two equations be

$$A^k + a_{k-1}A^{k-1} + \dots + a_oI = O \qquad (1)$$

and

$$A^k + b_{k-1}A^{k-1} + \dots + b_oI = O \qquad (2)$$

If these two equations are different, then not all of the numbers $a_{k-1} - b_{k-1}$, $a_{k-2} - b_{k-2}$, ... , $a_o - b_o$ are equal to 0. Let $a_j - b_j$ be the first of these numbers (reading from left to right) which is not 0. The letter j surely cannot be 0 for, if this were the case, all of the numbers $a_{k-1} - b_{k-1}$, $a_{k-2} - b_{k-2}$, ... , $a_1 - b_1$ would equal zero and $a_o - b_o \neq 0$. Then subtracting equation (2) from equation (1) would give

$$(a_o - b_o)I = O$$

or

$$I = O .$$

Such a result is absurd, so j must be one of the numbers k-1, k-2,... , 1. That is, $0 < j < k$. Now let us subtract equation (2) from equation (1) and divide by $a_j - b_j$. The result is

$$A^j + \frac{a_{j-1} - b_{j-1}}{a_j - b_j} A^{j-1} + \frac{a_{j-2} - b_{j-2}}{a_j - b_j} A^{j-1} + \dots + \frac{a_0 - b_0}{a_j - b_j} I = 0$$

This contradicts our assumption that A satisfies no equation of degree less than k. Therefore, assuming that there are two different least equations satisfied by A leads to a contradiction.

This uniqueness theorem, with the earlier existence statements, assures us that every matrix satisfies one and only one least equation. Since there is just one least equation staisfied by each matrix, it follows that each matrix (for which an inverse exists; i.e., for which $a_0 \neq 0$) has just one inverse. Thus a matrix cannot have two different inverses. Also, if A and B have the same inverse, then A = B.

Theorem 3.5 : If A has an inverse, then so does A^{-1}. In fact, the inverse of A^{-1} is A. In symbols, $(A^{-1})^{-1} = A$.

Proof: Let A be a matrix and let A^{-1} be its inverse. Consider now the equation $A^{-1}X - I = 0$. This is a polynomial equation in A^{-1} and $a_0 \neq 0$ so A^{-1} has an inverse. Adding I to both sides gives $A^{-1}X = I$, which makes it obvious that X = A. This means that the inverse of A^{-1} is A.

Theorem 3.6 : Let A and B be matrices of the same size. If A and B both have inverses, then so does their product AB. In fact, $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : Consider the equation $(AB)X - I = 0$. This is a polynomial equation satisfied by the matrix AB and $a_0 \neq 0$, so $(AB)^{-1}$ exists. Adding I to both sides of the equation gives

$$(AB)X = I\ .$$

By inspection, $X = B^{-1}A^{-1}$; therefore the inverse of AB is $B^{-1}A^{-1}$.

The reader may realize that an analogous theorem is true for the product of any finite number of matrices. Suppose $A_1, A_2, A_3, \ldots A_n$ are all (square) matrices of the same size and suppose their corresponding inverses are $A_1^{-1}, A_2^{-1}, \ldots, A_n^{-1}$. Then $(A_1A_2A_3\ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\ldots A_1^{-1}$. This can be summarized verbally by saying : given a finite number of (square) matrices of the same size, each of which has an inverse, the inverse of the product of the matrices is equal to the product of their inverses written in the reverse order.

Problem Section 3.4

1. Given $A = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$, find A^{-1} and B^{-1}.

   Use these inverses to find $(AB)^{-1}$. Check your work by calculating AB and then multiplying AB by $(AB)^{-1}$ to get I.

2. Let $D = AB = \begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix}$ (A and B from Problem #1.)

   Make use of some of the other information which you found in problem # 1 to solve the equation

   $$DW = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$ for W.

3. Let $A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$. Show that $\left((A^{-1})^{-1}\right)^{-1} = A^{-1} = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$.

4. If C satisfies the equation $C^2 - 2C + 3I = 0$, prove that C^{-1} exists and is $\frac{1}{3}(2I + C)$.

5. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$.

Show that the matrix

$$G = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}$$ is equal to AB.

Now solve the equation $GX = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ in two different ways.

The more direct method will be to calculate G^{-1} at once. The other method will be to obtain A^{-1} and B^{-1} and thereby obtain G^{-1}.

6. Let A and B be n x n matrices and assume B^{-1} exists. Show that

$BA^kB^{-1} = (BAB^{-1})^k$ where k is any positive integer.

7. Again, let A and B be n x n matrices and assume also that B^{-1} exists. Show that if the polynomial

$Y^k + a_{k-1}Y^{k-1} + \dots + a_0 I = 0$ is satisfied by A, then it is also satisfied by BAB^{-1}. What does this tell you about the number of solutions of such a polynomial?

TEST Q

1. **What is a matrix? Give an example of a matrix.**

2. **How are matrices added? How are they subtracted?**

   (a) Make up two matrices, call them A and B, and find

   (i) their sum: A + B

   (ii) their difference: A - B

3. **How are matrices multiplied?**

   (a) Make up two matrices, call them C and D, and find

   (i) their product: C D

APTITUDE TEST

1. Consider the following array of numbers:

$$\begin{array}{cccc} 1 & 2 & -1 & 4 \\ 3 & 0 & -2 & -5 \\ 5 & -3 & -4 & 6 \\ 8 & -6 & 7 & -7 \end{array}$$

We shall let A denote this array and the symbol $[A]_{i,j}$ shall mean the number which is in row i and column j. What is $[A]_{3,2}$? What is $[A]_{2,3}$? __________ __________

2. If such an array has six rows and six columns, how many numbers does it contain? If the number of rows were n and the number of columns were n, how many numbers would it contain? __________ __________.

3. Add the following:

(a) $1/2 + 3/4 + 5/8 =$ __________

(b) $1/x + 1/y + 1/z =$ __________

(c) $(x + y) + (-x/2 + y/3) - (-2z/3 + 4x) =$ __________

4. Multiply the following:

(a) $(2)(-1)(-3/2) =$ __________

(b) $(-a/b)(b/a)(c) =$ __________

(c) $(a + x)(-a - 3x) =$ __________

(d) $a + x(-a - 3x) =$ __________

(e) $(a - x)^2 (a + 3x) =$ __________

(f) $(a^2 - x^2)(a + x)^2 =$ __________

5. Subtract the sum of $a + 2b - c$ and $2a - 2b + 3c$ from the number 0.

____

6. Perform the indicated operations:

   a. $5(.60) + 10(.72) + 50(5.5) =$

   b. $(1/2)\ (2/3) = (1/3)\ (4/5) - (1/8)\ (4/3) =$

7. Solve each of the following equations for the letter x.

   a. $-2x + 3 = 7$

   b. $16 - 2x/3 = 5x + 4$

   c. $\dfrac{x + 2}{4} = \dfrac{5 - \frac{x}{2}}{3}$

   d. $ax - by = c$ ($a \neq 0$; that is, a does not equal 0)

   e. $2x/3 - 5 + 7/2 + 15 = -5x/8 + 4$

   f. $-3(8x/3 + 4a/3) - (2/3)\ (5a/4 + 3x/4) = (-3/2)(b - 4c)$

8. Find the roots of the following equations:

   a. $x^2 - 9 = 0$

   b. $x^2 + 6x + 9 = 0$

   c. $x^2 - 18 = 3x$

   d. $3x^2 - 7x/2 + 1 = 0$

9. Suppose the letters w and z are not comutative under multiplication. That is, suppose $wz \neq zw$. How would you expand the following?

   a. $(w + z)^2 =$

   b. $(w-z)\ (w + z) =$

10. Calculate the following:

(i) $(1/2)^4 =$

(ii) $(ab)^4 =$

(iii) $(a/b)^4 =$

(iv) $((ab)^2)^4 =$

(v) $a^7a^5 =$

(vi) $a^7 + a^5 =$

(vii) $(a^7 + a^5)^2 =$

11. Show that $a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \ldots\ldots + 1)$

12. Given $a^{-1} = 1/a \quad (a \neq 0)$ **, find:**

(i) $aa^{-1} =$

(ii) $(aa^{-1})^{-1} =$

(iii) $\dfrac{1}{a^{-1}} =$

(iv) $\dfrac{1}{(ab)^{-1}} =$

13. Let a and b be any numbers. If $ab = 0$, what can you conclude about a and b?

14. Let $5x^4 + 3x^2 - 2x + 4$ be a polynomial in x. Evaluate the polynomial for $x = 0$. For $x = -1$. For $x = ay$.

15. Given: $a^2 = 2a + 2$; Find: a^8 . Reduce the expression you obtain for a^8 to as simple a form as possible.

16. By algebraic manipulation transform the equation $a^4 + 5a^3 - 6a^2 + 7a + 5 = 0$ into the form $a(\qquad\qquad) = 1$.

ACHIEVEMENT TEST

1. Define the word "matrix."

2. How many elements are there in a matrix of size n?

3. Define equality of matrices.

4. Let $A = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$. Make use of the symbolism used to denote the elements of a matrix to give a concise description of the matrix A.

5. Express the system

$$\begin{aligned} 2x + y &= 7 \\ x - 3y &= 4 \\ a + b &= 6 \\ a - b &= 2 \end{aligned}$$

as a matrix equation.

6. Specify a non-zero 3 x 3 matrix A satisfying $[A]_{i,j} = -[A]_{j,i}$.

____

7. Perform the indicated operations:

(a) $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix} =$ ______________________________

(b) $\begin{bmatrix} 6 & 7 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} =$ ______________________________

(c) $5 \begin{bmatrix} 2 & 1 \\ 3 & 8 \end{bmatrix} =$ ______________________________

(d) $3 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} =$ ______________________________

(e) $2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} =$ ______________________________

8. Prove the following: Let A and B be two n x n matrices. Then A + B = B + A.

9. Solve 2(A + X) = 3(A + B) for the matrix X, given

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

____

10. Solve $3A - \frac{1}{2}X = -\frac{2}{3}(C - D)$ for the matrix X, given

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 8 & 8 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -3 & -2 \\ -5 & 0 \end{bmatrix}$.

____

11. Perform the indicated multiplications:

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} =$

(c) $\begin{bmatrix} -1 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \\ -2 & 2 & 1 & 2 \\ -3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 4 & 2 & 4 & 2 \\ -1 & 0 & 0 & -2 \end{bmatrix} =$

12. Given $C = \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, show that $CD \neq DC$

13. Given: $AB \neq BA$. Is the expression $A^2 - B^2$ equal to $(A-B)(A+B)$? Why?

14. Let A denote a 3 x 3 matrix. Prove that $AI = A$ for all A.

15. Let $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find D^2, D^3, and D^{174}.

16. In ordinary algebra $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both). In matrix algebra, however, this law breaks down. Devise an example which illustrates this.

17. Show that the matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A - I = O$

18. Given A and B are matrices of the same size, rewrite the product

$AB^4AA^2B^5BA^2AB^3B$ in a more condensed form.

19. Show that $A = \begin{bmatrix} 1 & 2x \\ 2/x & 1 \end{bmatrix}$ $(x \neq 0)$ satisfies the equation $B^2 - 2B - 3I = O$

How many solutions does this quadratic equation have? Why?

20. Show that $A^n - I = (A - I)(A^{n-1} + A^{n-2} + \dots + I)$

21. Let $A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -3 & 2 & -1 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ Show that $A^3 = I$. Find A^{124}.

22. Find two square roots of the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

23. Prove that there is no matrix X satisfying the equation

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}.$$

24. Given $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, solve the equation $YA = \begin{bmatrix} 16 & 32 \\ 20 & 8 \end{bmatrix}$ for Y.

25. Given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ find A^{-1}. Use this information to solve the equation $XA = \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}$ for X.

26. Find the least equation satisfied by the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and use the least equation to find A^{-1}.

27. The least equation satisfied by the matrix $A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$ is $A^2 - 5A + 4I = O$. Use this information to find A^4.

28. Find the least equation satisfied by $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Use the least equation obtained to compute B^{-1}.

29. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Prove that $A^{-1} \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$

(where $ad - bc =\neq 0$)

30. Given the least equation satisfied by a matrix how can one quickly determine whether or not the matrix has an inverse?

31. If B satisfied the equation $B^2 - 2B + 5I = O$, prove that $B^{-1} = \frac{1}{5}(2I - B)$.

32. Given $A^k + a_{k-1}A^{k-1} + \ldots + a_o I = O$, find an expression for A^{-1}.

33. Let A, B, C, and D be matrices of the same size and assume that A^{-1}, B^{-1}, C^{-1}, and D^{-1} exist. What is the inverse of the product ABCD?

34. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Use this information to devise an example which illustrates that in matrix algebra there is both a "right-hand" and a "left-hand" division.

35. Prove that the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is its own inverse. Use this fact to solve

$$XA = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{for } X.$$

____

36. Find the least equation satisfied by $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. What is A^{-1}?

____

QUESTIONNAIRE

1. **Did you have any experiences with programmed materials prior to this experiment?**

2. **If your answer to question #1 is 'yes', describe the materials and your reaction to them.**

3. **What is your reaction to the materials which you have just completed?**

   (a) Strong points

   (b) Weak points

4. **How do these materials compare with other programmed materials you have used?**

5. **How does your experience with these materials compare with other experiences you have had in the area of mathematics?**

6. **List any other comments here. These comments may pertain to the experiment as well as to the materials used in the experiment.**

APPENDIX VI

A. FORTRAN INSTRUCTION MATERIALS

B. FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - PROGRAMMER'S APTITUDE TEST

Pre-Test

1. **Write FORTRAN expressions equivalent to the following mathematical expressions.**

   a) $a + \frac{c}{d}$ ________ b) $2x - y$ ________

   c) $\frac{a}{(b)\ (c)}$ ________ d) r^5 ________

   e) r^{-5} ________

2. **State whether the mode of each the following quantities is floating point or fixed point.**

   a) I ________ b) X ________

   c) 2.0 ________ d) 327 ________

   e) 4.25E + 13 ________

3. **Suppose the current values of X, Y, I, and N are 2, 3, 4, and 17 respectively when control reaches each of the following arithmetic statements. Explain briefly what the effect will be.**

   a) Y = X - 10.0 ________

   b) I = X ________

   c) X = X + 1 ________

   d) X = N ________

4. **Write a statement that would cause control to transfer from itself to the statement numbered 36.**

   ____

5. **If the value of X were 17.5, what would happen when control reached the following statement:**

   IF(X-20.0)41, 50, 72 ________

6. What statement must be the last statement of every FORTRAN program? ______________________________

7. Suppose that at the time control reaches the following sequence of statements the value of N is 4., What will the value of SUM be when control exits from the DO loop?

```
    Y = 3
    SUM = 0.
    DO  25   I = 1, N
    SUM = SUM + Y
25  CONTINUE
```

FORTRAN FACTS TEST

Examination Part 1

(1) In the FORTRAN expression:

`1.0 - 6.0*D**(2+1)/DNUM`

which operation will be performed first?

a) Addition

b) Subtraction

c) Multiplication

d) Division

e) Exponentiation

f) There is no specific order

(2) Which one of the following is a Valid Floating Point Variable?

a) 48.236

b) NUM

c) DATE

d) 6.1719E-3

e) 31415

(3) Which of the following is a valid FIXED point variable?

a) NM-1

b) FIXED

c) MARRIED

d) IDATA(5)

e) None of these

(4, 5, 6) The array DELTA is communicated under the control of statement 25 which looks as follows:

```
25 FORMAT(11x, 12F10.6)
```

Answer the following questions about the printer output.

(4) Each print line would contain ________ elements.

a) 11
b) 10
c) 6
d) 12
e) As many elements per line as is permitted by the width of the paper.

(5) The first element of each new line will always be preceeded by ___________ spaces.

a) 11
b) at least 10
c) no more than 10
d) 10
e) none of these

(6) If the maximum element of the DELTA array were 25.68954 and the minimum were -8.632487 what would be the least number of blank spaces between successive elements on the same print line?

a) 2
b) 3
c) 1
d) 4
e) None of these

(7) Which of the following expressions is illegal because of mixed mode?

a) FIX(N, M,)+SUMX

b) ALPHA**(2*K)+TOTAL

c) JACK*ISSAC/KEATH

d) SCORE/(N-SCORE)

e) None of these.

(8) This statement will branch to _______ if N is 12 and K is 15.

a) Statement 5
b) Statement 6
c) Statement 7
d) None

```
IF(N-K)5,6,7
```

(9) If ALPHA is equal to 2.985, what will the value of MAX be after the execution of this statement

```
MAX = 1.2*ALPHA
```

a) 3.5870

b) 2

c) 3.0000

d) 3

e) None of these

(10) If K is equal to 3, the program will branch to statement number ______ as a result of executing this statement.

```
GO TO (25,27,19,13),K
```

a) 27

b) 3

c) 19

d) None of these

(11) Which of the following is not an acceptable way of writing the number 3.1416 as a constant?

a) 3.1416

b) 31416.**(-4)

c) .31416E1

d) 314.16E-2

(12) The variable IDENT has a value of 125 when the following instructions are performed:

```
      WRITE OUTPUT TAPE 6,15,IDENT
   15 FORMAT(I3)
```

(a) What would be printed? (b) Where would it be printed with respect to the previous line of printing?

a) ______________________________

b) ______________________________

(13) List in the order of printing the elements communicated by this output statement. K = 4

```
WRITE OUTPUT TAPE  6,7((BETA(I,J,K),J=1,2),I=1,3)
```

____

(14) In this program:

```
         DIMENSION  X(12),Y(15),Z(20)
         READ INPUT TAPE 5,15,X,Y,Z
   15    FORMAT (5X,8F8.5)
```

How many cards would be necessary to satisfy the input list?

________________________ 4

(15) In this program:

```
     DIMENSION DATA (2,2,2)
     READ INPUT TAPE 5,12,DATA
12   FORMAT (F10.4)
```

(a) How many card records will be read?
(b) In what order must the elements of the DATA array be presented . . . . List the elements in order.

a) ______________________________

b) ______________________________

(16) Describe how the information would be punched on cards to correspond to this Input and Format statement:

```
      READ  100,ID,N,X,Y,Z
  100 FORMAT  (I 3, 5X, I2, 4(5X,F10.5))
```

Indicate in what columns on the Input card that each variable must be punched and where the decimal point is assumed to be. Make your answer on the attached IBM card.

(Attach IBM card here)

(17) Write a FORMAT specification which will cause the printing of the following heading at the beginning of a new page.

bbbbbXbVALUEbbbbbYbVALUEbbbbbANSWER

(the small letter b stands for a blank space.)

(18) Write the FORTRAN control statement to accomplish the following:

Branch to Statement #5 if K greater than N

Branch to Statement #6 is K equals N

Branch to Statement #7 if K less than N

(19) **Write the FORTRAN control statements to accomplish the following:**

Branch to Statement #20 if K greater than N and K less than M

Branch to Statement #25 if K greater than N and K greater than or equal to M

Branch to Statement #30 is K less than or equal to N and K greater than 2

Branch to Statement #35 if K less than or equal to N and K less than or equal to 2

(20) **Write a DO Loop sequence which will set the first K elements of the Z array equal to the corresponding element in the W array.**

(21) **Write an arithmetic statement to evaluate this mathematical formula.**

$$REC = \frac{ax^2+bx+c}{3x+F}$$

Each of the following arithmetic statements are not acceptable statements . . . . . why?

(22) P1*V2/T1=P2*V2/T2

(23) INDEX = L**2.+M

(24) 3.1416=PHI

PROBLEM EVALUATION TEST

Examination Part 2

This problem requires the generation of a Monthly Loan Payment/Interest Table. The input, shown on a separate page, contains the necessary information to perform the computation and subsequent table entry outputs.

Formulae for this problem are as follows:

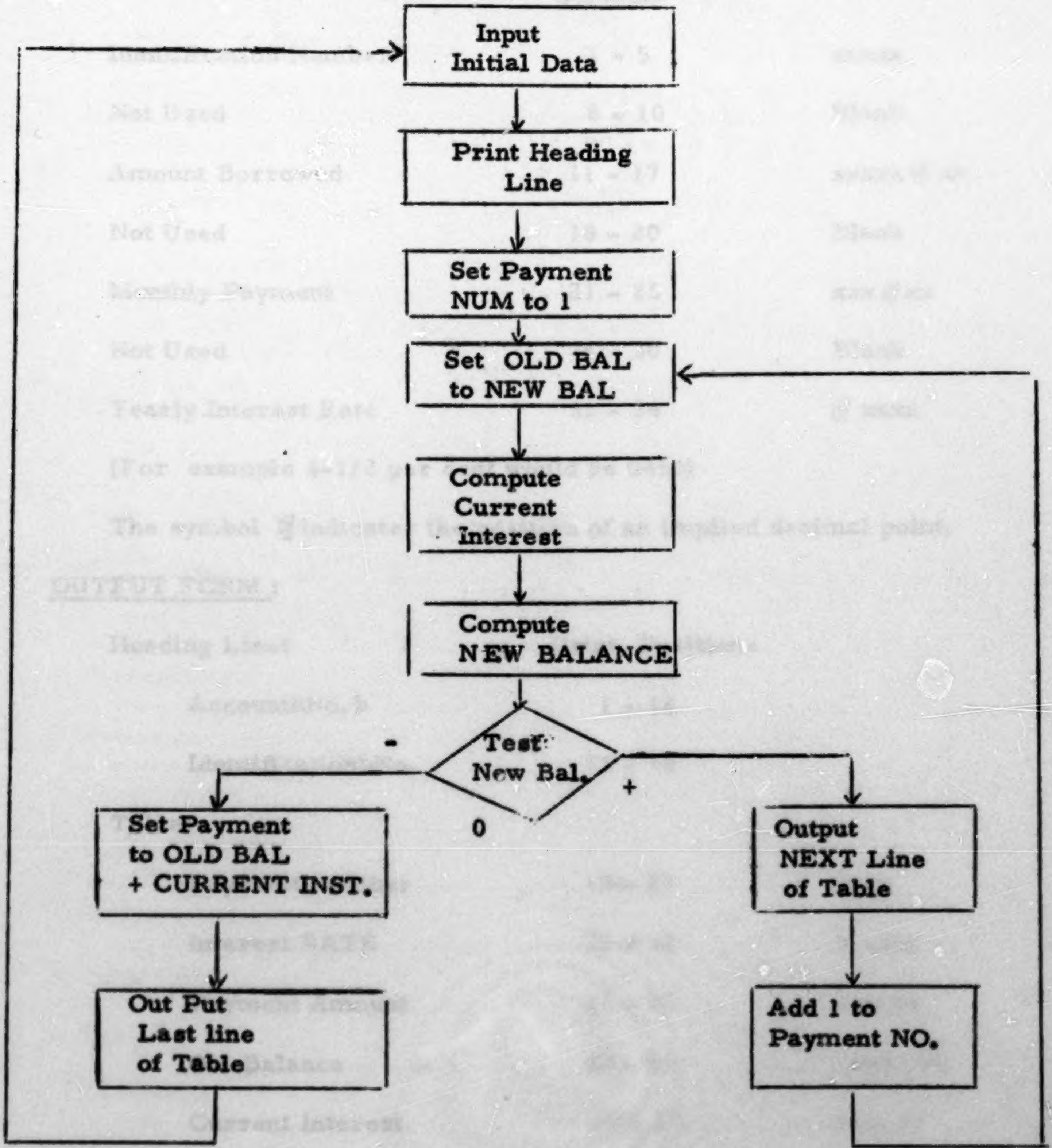
Current Interest = Old Balance X RATE/12

New Balance = Old Balance + Current Interest - Payment

Use the Flow Diagram as a guide.

Note: The original amount borrowed may be considered the same as the Old Balance at the time of the first payment. Also, the last payment may be less than the normal payment, for example if the New Balance became negative, the last payment must be changed to Old Balance plus Current Interest.

FLOW DIAGRAM


Input Initial Data
Print Heading Line
Set Payment NUM to 1
Set OLD BAL to NEW BAL
Compute Current Interest
Compute NEW BALANCE
Test New Bal.
-
+
0
Set Payment to OLD BAL + CURRENT INST.
Out Put Last line of Table
Output NEXT Line of Table
Add 1 to Payment NO.

INPUT FORM:

	Columns	
Identification Number	1 - 5	xxxxx
Not Used	6 - 10	Blank
Amount Borrowed	11 - 17	xxxxx @ xx
Not Used	18 - 20	Blank
Monthly Payment	21 - 25	xxx @ xx
Not Used	26 - 30	Blank
Yearly Interest Rate	31 - 34	@ xxxx

(For example 4-1/2 per cent would be 0450)

The symbol @ indicates the position of an implied decimal point.

OUTPUT FORM :

Heading Line:	Print Positions	
AccountbNo.b	1 - 12	
IdentificationbNo.	13 - 18	
Table:		
Payment Number	18- 21	xxxx
Interest RATE	25 - 30	0.xxxx
Payment Amount	36 - 41	xxx.xx
Old Balance	47 - 54	xxxxx.xx
Current Interest	60 - 65	xxx.xx
NEW Balance	71 - 78	xxxxx.xx

The small "b" in the Heading Line indicates that a blank space is to be left.

EVALUATION OF THE FORTRAN SELF-INSTRUCTIONAL PROGRAM

I want to take this opportunity to thank you for participating in the evaluation of the instructional materials and methods of presentation used in this series. Your personal comments and advice regarding this project will serve as a guide for revision and improvement.

We would like your comments in three major areas.

1. Content of the instruction
2. Method of presentation
3. How may we improve this series for future High School Science Institute groups.

Content Evaluation

Topics	Not Enough	About Right	Too Much
Arithmetic statement general			
Operations Symbols			
Hierarchy rules			
Use of parentheses			
Constants			
Variables			
Arithmetic modes			
Subscripts			
Expressions			
E-Notation			
Control statements general			
Statement numbering			
IF statement			
GO TO statement			
DO statement			
Continue statement			
Computed GO TO			

Topics	Not Enough	About Right	Too Much
Input/Output general			
Dimension statement			
Tape Input			
Tape Output			
FORMAT statement			
F type conversion			
I type conversion			
E type conversion			
H type conversion			
Spacing control			
Problem Organization General			
Flow Diagraming			
Planning for Input/Output			

Personal Comments:

Method of Presentation:

In which group did you participate? (please check one)

___ Individual slide-lecture presentation

___ Group slide-lecture presentation (med ctr)

___ IBM programmed text

How did you like this method of presentation?

How may it be improved?

Do you feel that the FORTRAN OUTLINE is a valuable reference?

Would the course have been improved if references had been made to the outline during each instructional session? ___

What suggestion would you make for the presentation of a FORTRAN series for future High School Science Institute?

Any other comments:

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES Summer '64

TABLE OF POSITIVE EIGENVALUES

No.	Eigenvalue	Percent of Communality Over All (83) Factors		20 Rotated Factors	
1	8.001	9.6	9.6	12.8	12.8
2	4.875	5.9	15.5	7.8	20.6
3	4.094	4.9	20.4	6.5	27.1
4	3.978	4.8	25.2	6.4	33.4
5	3.812	4.6	29.8	6.1	39.5
6	3.569	4.3	34.1	5.7	45.2
7	3.365	4.1	38.2	5.4	50.6
8	3.147	3.8	42.0	5.0	55.6
9	3.062	3.7	45.7	4.9	60.5
10	2.982	3.6	49.3	4.8	65.3
11	2.760	3.3	52.6	4.4	69.7
12	2.691	3.2	55.8	4.3	74.0
13	2.585	3.1	58.9	4.1	78.1
14	2.494	3.0	61.9	4.0	82.1
15	2.278	2.7	64.7	3.6	85.7
16	1.996	2.4	67.1	3.2	88.9
17	1.958	2.4	69.5	3.1	92.0
18	1.730	2.1	71.5	2.8	94.8
19	1.667	2.0	73.5	2.7	97.5
20	1.593	1.9	75.5	2.5	100.0
21	1.468	1.8	77.2		
22	1.423	1.7	78.9		
23	1.380	1.7	80.6		
24	1.318	1.6	82.2		
25	1.258	1.5	83.7		
26	1.208	1.5	85.2		
27	1.116	1.3	86.5		
28	1.009	1.2	87.7		

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - Summer 1964

Rotated Factor Matrix

		Factor Numbers	11	12	13	14	15	16	17	18	19	20
Sum Squares Over Variables			3.077	2.840	2.300	2.484	2.969	2.856	2.688	2.691	2.723	2.476
Variable No.	Variable Name	Communality 20 Factors										
1	I- 1	0.822	0.094	-0.024	-0.130	-0.113	0.093	0.840	-0.020	0.097	0.056	0.079
2	I- 2	0.827	0.027	-0.077	-0.017	-0.146	0.151	-0.058	-0.007	0.098	-0.032	-0.227
3	I- 3	0.578	-0.002	-0.025	0.266	0.057	0.311	0.104	0.007	0.004	0.071	0.259
4	I- 4	0.818	-0.037	-0.249	-0.218	0.039	0.247	0.005	0.024	-0.067	-0.024	0.004
5	I- 6	0.868	-0.844	-0.043	0.044	-0.147	-0.040	0.113	0.032	0.053	-0.019	-0.012
6	I- 7	0.797	-0.326	-0.041	0.178	-0.083	0.090	-0.027	0.164	0.184	-0.008	0.350
7	I- 8	0.835	0.176	0.139	-0.020	-0.127	0.041	-0.082	-0.054	-0.107	-0.120	0.220
8	I- 9	0.786	-0.180	-0.030	-0.101	-0.058	-0.200	-0.021	-0.013	0.036	0.138	-0.078
9	I-10	0.879	0.025	0.023	-0.061	0.078	-0.075	-0.073	0.896	0.027	0.091	0.026
10	I-11	0.696	-0.046	0.102	0.019	0.202	-0.029	0.000	0.206	-0.016	0.741	-0.037
11	I-12	0.773	-0.540	0.093	-0.335	0.104	-0.056	-0.078	-0.122	0.297	0.249	0.046
12	I-13	0.773	-0.053	0.163	-0.280	0.134	0.239	0.006	-0.074	-0.003	-0.056	0.107
13	I-14	0.752	0.070	0.145	0.028	-0.082	-0.016	-0.093	0.076	0.064	-0.014	-0.017
14	I-15	0.905	-0.229	0.078	-0.040	0.164	0.107	-0.094	0.004	0.046	0.595	0.191
15	I-16	0.733	-0.117	-0.277	-0.044	-0.263	-0.164	-0.064	-0.034	0.027	0.408	0.132
16	I-17	0.794	0.177	0.100	0.041	0.083	-0.026	0.007	-0.148	0.007	0.261	-0.219
17	I-18	0.717	-0.200	-0.206	-0.049	-0.134	-0.049	-0.044	0.024	-0.277	0.025	0.070
18	I-19	0.943	-0.149	-0.036	-0.144	0.035	0.079	-0.019	-0.053	-0.025	0.052	-0.048
19	I-20	0.892	-0.124	-0.025	-0.004	0.033	0.170	0.016	-0.042	0.081	0.017	0.008
20	I-21	0.852	-0.108	-0.006	0.055	0.150	-0.074	-0.005	0.083	0.002	0.157	-0.003
21	I-22	0.802	-0.071	-0.275	0.090	0.057	-0.036	0.027	0.256	-0.237	0.271	0.082
22	I-23	0.810	0.175	-0.142	0.137	0.366	-0.059	-0.120	0.150	-0.246	-0.028	0.294
23	I-24	0.760	0.179	-0.066	0.039	0.320	0.077	0.195	-0.145	-0.055	0.213	0.125
24	I-25	0.759	-0.132	-0.022	0.031	-0.093	0.023	-0.266	-0.084	0.014	0.081	0.059
25	I-26	0.798	0.018	-0.082	-0.183	-0.141	0.082	0.188	0.117	-0.034	-0.017	0.013
26	II-29	0.841	0.095	-0.634	-0.167	0.197	0.204	-0.033	-0.031	0.088	-0.027	-0.289
27	II- 30	0.794	0.058	0.113	0.007	-0.027	-0.169	-0.037	-0.054	0.163	0.007	0.100
28	II-31	0.661	-0.136	-0.007	0.093	0.249	0.231	-0.012	0.022	-0.118	-0.107	0.161

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - SUMMER 1964

Rotated Factor Matrix

Variable No.	Variable Name	Factor Numbers / Communality 20 Factors	11	12	13	14	15	16	17	18	19	20
Sum Squares		Over Variables	3.077	2.840	2.300	2.484	2.969	2.856	2.688	2.691	2.723	2.476
29	II-32	0.818	-0.100	-0.113	-0.120	-0.045	-0.150	0.055	0.086	-0.091	-0.114	0.024
30	II-33	0.660	0.099	-0.081	0.098	-0.088	-0.099	0.016	-0.035	0.009	-0.238	0.186
31	II-34	0.737	0.031	-0.128	-0.022	-0.044	0.200	-0.047	-0.021	-0.031	-0.051	-0.787
32	II-35	0.785	-0.266	0.077	0.120	0.104	-0.018	0.148	0.428	-0.058	-0.113	-0.217
33	II-36	0.747	0.067	0.126	-0.133	-0.035	0.151	0.022	0.009	0.099	-0.116	-0.003
34	II-37	0.827	-0.027	0.328	0.051	-0.072	0.315	0.070	0.003	0.067	-0.176	-0.341
35	II-38	0.721	-0.026	0.002	-0.143	0.221	-0.288	-0.023	-0.150	-0.242	-0.068	0.020
36	II-39	0.845	-0.489	0.117	0.167	-0.112	0.026	-0.053	0.025	-0.023	0.054	-0.224
37	II-40	0.754	-0.122	-0.299	0.019	-0.084	-0.079	0.002	-0.094	0.130	0.161	0.059
38	II-41	0.749	-0.055	0.016	-0.141	-0.041	-0.016	-0.084	0.012	-0.015	-0.055	0.142
39	II-42	0.673	0.126	0.061	-0.010	-0.106	-0.334	0.323	0.248	0.058	-0.223	-0.077
40	II-43	0.739	-0.048	0.005	-0.035	0.265	0.194	-0.143	0.138	-0.203	0.180	-0.227
41	II-44	0.698	-0.160	-0.202	-0.072	0.296	0.029	-0.046	-0.078	0.183	-0.181	-0.134
42	II-45	0.471	0.088	-0.027	-0.125	0.551	0.000	-0.013	0.109	-0.037	0.199	-0.017
43	II-46	0.740	-0.003	0.078	-0.018	-0.178	-0.137	0.201	-0.130	0.043	0.142	0.054
44	II-47	0.590	-0.205	-0.361	-0.045	-0.321	-0.020	-0.086	0.095	0.242	-0.024	0.119
45	II-48	0.715	-0.106	-0.584	-0.074	-0.054	-0.250	0.064	-0.055	-0.051	-0.220	-0.080
46	II-49	0.790	0.005	0.162	-0.018	0.042	-0.092	-0.005	0.220	0.059	-0.113	-0.014
47	II-50	0.786	-0.139	-0.044	0.135	-0.104	-0.084	0.182	0.068	0.101	-0.051	0.107
48	II-51	0.717	-0.277	-0.107	0.040	0.061	0.158	0.321	0.402	0.267	0.001	-0.010
49	II-52	0.753	-0.215	-0.010	-0.379	0.110	0.042	0.213	-0.122	-0.101	-0.011	0.183
50	II-53	0.695	0.079	-0.001	-0.042	0.173	-0.126	0.146	-0.025	-0.123	0.077	0.095
51	II-54	0.755	0.203	-0.002	-0.038	-0.384	0.311	0.073	0.034	0.035	-0.005	0.431
52	II-55	0.660	0.106	0.407	0.117	0.121	0.040	0.313	0.079	-0.099	-0.021	0.099
53	II-56	0.474	0.116	0.236	0.080	0.131	-0.213	0.173	-0.021	-0.020	-0.012	-0.117
54	II-57	0.744	-0.056	0.073	-0.007	-0.023	-0.805	-0.087	-0.032	0.021	0.003	0.120
55	II-58	0.656	0.054	0.070	-0.021	-0.017	-0.080	0.195	0.083	0.076	-0.131	0.131
56	II-59	0.616	0.054	-0.035	-0.028	0.065	-0.587	0.115	0.179	0.132	0.111	0.110
57	II-60	0.764	0.037	0.105	-0.056	-0.089	0.068	-0.246	-0.026	-0.713	0.061	-0.041
58	II-61	0.702	-0.095	0.090	-0.248	0.105	-0.166	0.166	0.011	-0.229	-0.148	0.235
59	II-62	0.742	-0.327	0.064	-0.021	0.155	0.002	-0.063	-0.015	-0.046	-0.110	0.267
60	II-63	0.703	-0.090	0.650	-0.085	0.039	-0.239	-0.073	-0.028	-0.175	-0.013	-0.017

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - Summer 1964

Rotated Factor Matrix

		Factor Numbers	11	12	13	14	15	16	17	18	19	20
Sum Square		Over Variables	3.077	2.840	2.300	2.484	2.969	2.856	2.688	2.691	2.723	2.476
Variable		Communality										
No.	Name	20 Factors										
61	II-64	0.668	0.198	-0.082	-0.061	-0.026	-0.299	0.054	-0.300	0.291	0.034	0.349
62	II-65	0.698	0.045	-0.074	-0.763	-0.039	-0.017	0.037	0.041	0.061	-0.027	-0.055
63	II-66	0.790	-0.059	0.170	0.127	-0.049	-0.528	-0.163	0.125	0.122	-0.180	0.119
64	III-67	0.888	0.018	-0.004	0.057	0.063	-0.062	-0.003	0.749	-0.024	0.156	0.051
65	III-68	0.899	-0.552	-0.143	0.165	-0.106	-0.004	0.015	0.070	0.167	0.292	0.092
66	III-69	0.678	0.063	0.062	0.118	0.702	0.000	-0.101	0.076	0.082	0.062	0.071
67	III-70	0.742	-0.047	0.221	-0.290	0.025	-0.163	-0.063	0.194	0.050	-0.259	0.069
68	III-71	0.834	0.255	-0.065	-0.072	0.258	-0.048	0.383	-0.043	0.281	-0.166	-0.209
69	III-72	0.683	0.237	-0.161	0.149	0.046	-0.150	-0.038	0.104	0.068	-0.082	0.233
70	III-73	0.787	-0.135	-0.065	0.328	-0.041	-0.026	0.417	0.011	0.329	-0.086	0.166
71	III-74	0.799	-0.003	-0.031	0.154	-0.155	-0.010	-0.045	0.037	-0.007	0.048	-0.110
72	III-75	0.701	-0.146	-0.272	-0.081	0.076	-0.202	-0.226	0.036	0.534	-0.215	-0.029
73	III-76	0.721	-0.061	-0.181	-0.044	-0.045	-0.012	-0.091	-0.057	0.502	0.236	0.091
74	III-77	0.774q	-0.146	0.140	0.483	0.037	-0.043	0.089	0.133	0.133	-0.416	-0.069
75	III-78	0.803	-0.316	0.056	0.090	-0.022	-0.071	0.723	-0.041	-0.224	-0.096	-0.043
76	III-79	0.826	-0.287	-0.305	0.073	0.312	0.106	0.348	-0.131	0.026	-0.361	-0.175
77	III-80	0.786	-0.033	-0.011	0.107	-0.058	0.123	-0.151	0.289	0.034	0.116	0.088
78	III-81	0.710	0.056	-0.220	-0.146	0.062	0.113	0.201	0.055	-0.011	-0.122	0.046
79	III-82	0.805	0.049	0.175	0.211	0.034	0.239	-0.007	-0.045	0.070	0.020	0.119
80	III-83	0.702	0.070	0.134	0.170	0.073	0.052	-0.042	0.064	-0.023	-0.246	0.148
81	III-84	0.719	0.102	-0.063	0.273	-0.109	0.215	0.004	0.277	-0.065	-0.165	-0.004
82	III-85	0.772	-0.068	0.149	-0.220	-0.117	0.106	0.104	0.200	0.043	-0.146	-0.004
83	III-86	0.751	0.147	-0.122	0.067	0.221	0.044	0.217	-0.046	-0.577	-0.097	0.010

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - Summer 1964

		Factor Number	1	2	3	4	5	6	7	8	9	10
Sum	Square	Over Variables	5.888	3.828	2.455	3.620	3.895	3.161	3.266	2.881	3.518	3.018
Variable No.	Name	Communality 20 Factors										
1	I-1	0.822	-0.074	-0.071	0.032	0.059	0.159	-0.033	0.014	0.009	-0.081	0.047
2	I-2	0.827	0.080	0.008	0.070	-0.818	0.010	0.016	0.140	0.005	0.077	0.092
3	I-3	0.578	-0.089	0.315	0.071	0.126	0.037	0.082	0.039	-0.330	-0.172	0.219
4	I-4	0.818	0.081	-0.184	0.212	-0.269	-0.069	-0.089	0.353	-0.083	0.576	0.074
5	I-6	0.868	0.194	0.042	0.006	0.018	0.011	0.107	-0.187	0.122	0.090	0.051
6	I-7	0.797	0.490	-0.036	0.146	-0.301	-0.109	-0.001	-0.079	-0.265	0.054	0.117
7	I-8	0.835	0.579	0.063	0.195	0.091	-0.135	-0.410	-0.197	0.004	-0.053	-0.261
8	I-9	0.786	0.200	0.730	0.050	0.087	-0.028	-0.160	-0.099	0.037	0.113	-0.198
9	I-10	0.879	-0.063	-0.155	-0.005	0.024	-0.057	0.057	0.004	-0.006	-0.067	-0.066
10	I-11	0.696	-0.001	0.086	0.037	0.043	-0.145	0.012	0.022	-0.042	-0.097	0.065
11	I-12	0.773	0.156	0.083	0.138	0.022	-0.032	-0.054	-0.007	-0.135	-0.085	-0.306
12	I-13	0.773	0.225	0.149	-0.275	0.067	-0.070	0.261	-0.537	-0.127	0.163	0.113
13	I-14	0.752	-0.047	-0.080	0.056	-0.010	-0.025	0.105	0.785	0.094	0.156	0.161
14	I-15	0.905	0.391	0.054	0.184	-0.223	-0.118	0.007	-0.166	-0.010	0.307	0.177
15	I-16	0.733	0.190	0.046	0.083	-0.500	-0.058	0.102	0.095	0.175	0.062	-0.055
16	I-17	0.794	0.477	0.150	0.149	-0.341	-0.220	-0.314	0.169	-0.013	0.122	-0.164
17	I-18	0.717	0.521	-0.102	-01141	-0.022	-0.318	0.090	0.259	-0.161	0.018	-0.147
18	I-19	0.942	0.913	0.147	-0.046	0.045	-0.000	0.085	0.051	0.018	-0.041	0.109
19	I-20	0.892	0.893	0.105	-0.082	0.059	-0.023	0.044	0.043	-0.017	0.084	0.079
20	I-21	0.852	0.801	0.138	-0.159	-0.233	-0.025	0.063	-0.088	0.122	0.062	0.109
21	I-22	0.802	0.451	-0.000	-0.170	0.040	-0.418	-0.109	-0.153	-0.079	0.234	-0.033
22	I-23	0.810	0.242	-0.075	-0.164	-0.133	-0.486	0.058	0.041	0.027	0.173	0.196
23	I-24	0.760	0.259	-0.035	-0.028	0.292	-0.478	-0.084	0.052	0.281	0.053	0.130
24	I-25	0.759	0.156	-0.040	-0.088	-0.133	-0.738	-0.185	0.010	-0.033	-0.097	0.033
25	I-26	0.798	-0.001	0.147	0.100	-0.051	-0.729	0.133	-0.093	-0.184	0.009	-0.235
26	II-29	0.841	0.288	-0.116	0.055	-0.231	-0.024	0.029	0.245	0.088	0.048	0.060
27	II-30	0.794	-0.066	-0.090	-0.025	-0.784	-0.001	0.008	-0.162	-0.101	-0.101	0.183
28	II-31	0.661	-0.073	-0.141	-0.016	-0.144	0.581	-0.124	-0.049	0.115	-0.089	-0.209
29	II-32	0.818	-0.072	0.329	-0.009	0.012	-0.030	-0.770	-0.086	0.087	0.023	0.028
30	II-33	0.660	0.512	-0.277	0.172	0.081	-0.059	-0.055	-0.240	-0.244	-0.134	0.087
31	II-34	0.737	-0.050	-0.114	0.006	-0.024	0.008	-0.056	-0.060	-0.124	-0.110	-0.019
32	II-35	0.785	0.194	-0.246	0.163	0.090	-0.181	-0.055	-0.356	0.018	-0.073	0.336
33	II-36	0.747	0.194	-0.055	-0.781	0.017	-0.040	0.032	-0.025	-0.023	-0.012	-0.081

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - Summer 1964

Rotated Factor Matrix

Variable No.	Variable Name	Factor Number	1	2	3	4	5	6	7	8	9	10
Sum Squares		Over Variables	5.888	3.828	2.455	3.620	3.895	3.161	3.266	2.881	3.518	3.018
		Communality 20 Factors										
34	II-37	0.827	0.093	0.076	-0.159	-0.027	-0.032	0.074	-0.242	-0.005	0.580	-0.116
35	II-38	0.721	-0.214	0.025	-0.139	0.152	0.081	0.146	0.093	-0.145	0.560	0.147
36	II-39	0.845	0.103	0.206	-0.036	0.044	0.208	-0.423	0.166	-0.108	0.296	0.297
37	II-40	0.754	0.030	0.076	-0.104	-0.008	0.039	0.095	0.068	-0.000	0.739	-0.042
38	II-41	0.749	0.091	0.800	0.037	-0.156	-0.025	-0.061	-0.112	-0.029	-0.035	-0.011
39	II-42	0.673	0.030	0.182	0.168	-0.094	0.074	0.008	-0.442	-0.153	0.067	0.071
40	II-43	0.739	-0.062	-0.240	-0.023	-0.117	0.183	-0.583	0.016	0.003	0.031	-0.112
41	II-44	0.698	0.250	-0.121	0.240	0.024	-0.101	-0.330	0.091	0.106	0.278	0.308
42	II-45	0.471	-0.094	-0.083	-0.030	0.078	0.105	0.121	-0.047	-0.047	-0.155	-0.114
43	II-46	0.740	0.176	-0.196	0.005	-0.120	0.079	0.034	0.119	-0.056	-0.075	0.698
44	II-47	0.590	0.115	0.107	0.048	-0.072	-0.028	0.085	0.210	0.340	0.112	-0.099
45	II-48	0.715	0.350	0.124	0.066	-0.041	-0.020	-0.169	-0.133	-01092	0.115	-0.122
46	II-49	0.790	0.059	0.055	-0.171	-0.208	0.010	-0.031	-0.046	-0.770	0.092	0.066
47	II-50	0.786	0.227	0.226	0.056	0.157	0.605	-0.372	0.045	-0.093	-0.123	0.081
48	II-51	0.717	0.002	-0.094	0.192	-0.191	-0.194	0.128	0.006	-0.247	0.087	0.239
49	II-52	0.753	-0.166	-0.041	-0.300	-0.418	-0.062	-0.231	0.224	0.017	0.148	-0.142
50	II-53	0.695	0.201	0.470	0.035	-0.452	-0.196	-0.131	0.233	0.028	-0.020	0.106
51	II-54	0.755	0.021	-0.040	-0.152	0.125	-0.121	0.146	0.391	0.007	-0.088	0.191
52	II-55	0.660	0/079	0.004	0.200	-0.013	0.035	0.064	0.406	-0.171	0.112	-0.266
53	II-56	0.474	-0.128	-0.122	0.307	-0.181	0.281	0.079	0.135	-0.110	0.121	0.054
54	II-57	0.744	0.002	-0.013	0.105	-0.106	0.001	-0.076	0.113	0.039	-0.104	-0.096
55	II-58	0.656	0.021	-0.010	0.202	-0.212	0.089	-0.134	-0.040	-0.614	-0.137	0.217
56	II-59	0.616	-0.274	0.141	-0.027	0.084	-0.049	0.022	-0.053	-0.129	-0.087	0.207
57	II-60	0.764	-0.042	0.152	0.102	0.154	-0.120	-0.173	-0.124	-0.120	-0.100	0.135
58	II-61	0.702	0.010	0.132	-0.153	00.139	-0.254	-0.059	0.532	-0.082	-0.005	-0.095
59	II-62	0.742	0.116	-0.265	0.262	0.136	-0.475	-0.186	0.164	-0.200	0.882	-0.111
60	II-63	0.703	0.207	-0.074	-0.076	-0.148	-0.133	0.064	0.253	-0.021	-0.045	-0.066
61	II-64	0.668	0.139	0.003	-0.030	0.015	-0.136	-0.099	0.402	-0.122	0.039	-0.040
62	II-65	0.698	0.049	0.181	-0.102	-0.030	-0.221	0.007	-0.032	-0.008	-0.014	0.004
63	II-66	0.790	-0.109	0.191	0.166	0.100	-0.092	0.063	-0.024	0.501	0.067	0.036
64	III-67	0.888	-0.088	0.288	-0.037	0.070	-0.048	-0.374	0.130	-0.153	-0.104	0.076
65	III-68	0.899	0.189	0.430	-0.073	0.095	0.093	-0.022	0.242	-0.313	-0.087	-0.014
66	III-69	0.678	0.250	0.009	0.079	0.105	0.038	-0.104	0.004	-0.027	0.190	-0.056

FACTOR ANALYSIS OF FORTRAN PROGRAMMED LECTURES - Summer 1964

Rotated Factor Matrix

		Factor Number	1	2	3	4	5	6	7	8	9	10
Sum	Squares	Over Variables	5.888	3.828	2.455	3.620	3.895	3.161	3.266	2.881	3.518	3.018
Variable No.	Variable Name	Communality 20 Factors										
67	III-70	0.742	0.093	-0.408	-0.214	-0.095	-0.207	-0.131	0.096	-0.375	0.135	0.072
68	III-71	0.834	0.469	-0.053	-0.296	0.144	0.153	0.006	-0.050	-0.014	0.028	0.184
69	III-72	0.688	0.210	-0.252	-0.215	-0.309	-0.094	0.052	-0.105	-0.322	0.306	-0.110
70	III-73	0.787	0.189	0.236	0.092	0.163	-0.107	-0.381	0.123	-0.086	-0.099	-0.157
71	III-74	0.799	-0.147	-0.306	-0.031	-0.263	0.021	-0.452	-0.065	-0.357	-0.215	0.485
72	III-75	0.701	0.104	-0.074	0.077	-0.182	-0.0089	0.054	0.208	-0.049	-0.095	0.234
73	III-76	0.721	-0.031	-0.374	0.047	-0.092	-0.028	-0.146	-0.092	-0.005	0.333	0.284
74	III-77	0.774	-0.029	-0.015	-0.027	-0.361	-0.332	0.070	-0.135	-0.085	-0.010	-0.039
75	III-78	0.803	0.024	-0.029	-0.069	-0.038	-0.059	0.084	-0.113	-0.209	0.079	0.128
76	III-79	0.826	-0.049	0.272	0.212	0.087	-0.090	0.252	0.054	-0.077	-0.151	-0.068
77	III-80	0.786	0.192	0.229	0.031	-0.199	0.133	0.095	0.029	-0.268	0.149	0.612
78	III-81	0.710	0.021	-0.211	0.142	-0.120	-0.364	-0.186	-0.171	-0.049	-0.015	0.527
79	III-82	0.805	0.022	-0.187	0.084	0.010	-0.166	-0.247	0.063	-0.142	0.674	-0.197
80	III-83	0.702	0.335	0.231	0.193	0.009	-0.074	-0.077	-0.018	0.149	0.521	0.214
81	III-84	0.719	0.174	0.346	-0.261	-0.377	-0.243	-0.189	0.087	0.008	0.035	0.025
82	III-85	0.772	0.210	-0.296	0.431	-0.158	-0.079	0.401	0.033	-0.108	-0.261	-0.085
83	III-86	0.751	0.286	-0.112	0.352	0.078	-0.017	-0.126	-0.082	0.104	0.055	-0.081

APPENDIX VII

SYMBOLIC LOGIC INSTRUCTION MATERIALS

STUDY GUIDE FOR PRESENTATION II

Name ______________________

Time required ______ ______ ______

Answer each question, or solve the problem, in the space provided.

Check the blank at the left if you know the answer, or can solve the problem, without checking back in the instructional materials.

Check the blank at the right if it is necessary to look up the answer or look up how to solve the problem.

Leave both blanks unchecked if you cannot respond at all.

I. Answer questions 1 - 21 after completing the equivalence function
(MIDDLE OF PAGE 7)

	Did not Check Back	Checked Back
1. What is a proposition?	______	______
2. Are all assertions propositions?	______	______
3. Is the following a proposition? **John is 16 years old.**	______	______
4. In your own words, give a definition of 'logical constant'.	______	______
5. On what does the truth-value of the logical constants depend?	______	______

	Did not Check Back	Checked Back
6. Given the two propositions: (a) The sun is shining. (b) It is cloudy. Link these into a compound proposition using a logical constant	________	______
7. What is the truth-table for (p . q)? (last column)	________	________
8. Why is the conjunctive function false for all cases except one?	________	________
9. What is the truth-value of John is big and strong if it is false that John is big and true that John is strong.	________	________
10. What is the truth-table for (p v q)?	________	________
11. Why is the disjunctive function false for only one case?	________	________

	Did not check back	Checked Back
12. What is the truth-value of John is big or strong when it is false that John is big and true that John is strong?	________	________
13. What is the truth-table for $(p \supset q)$?	________	________
14. Why is the implicative function true when p and q are both true and also when p and q are both false?	________	________
15. Give an example of the third row of the truth-table for $(p \supset q)$?	________	________
16. When are two propositional functions considered logically equivalent?	________	________
17. The truth-tables for the two functions $(p \vee q)$ and $(q \vee p)$ are the same. Therefore these two functions are ________________	________	________

	Did not Check Back	Checked Back
18. Give two simple propositional functions which you would predict are logically equivalent?		
19. What is the truth-table for $(p \equiv q)$?		
20. Why is the equivalence function true when both members are false?		
21. Give a verbal instance of the equivalence function.		

II. Answer items 22 - 36 after completing Presentation .

22. The logical constants 'v', '.', etc. are called binary constants because they operate on whatever is on both sides of them. What constant is not a binary constant?		
23. The logical constants need help in linking propositional variables. What symbols provide this function?		

	Did not Check Back	Checked Back
24. Put the following propositional function into symbolic form using the appropriate symbols. If John is sick, then either he will send a note or call by telephone.	________	________
25. Truth tables provide a means of classifying arguments into three classes. What are these?	________	________
26. Arguments in which the conclusion contradicts one or both of the premises and arguments in which the conclusion may be true part of the time are both called invalid. Why are both of these arguments invalid?	________	________
27. Give an example of an argument when the conclusion logically follows from the premises.	________	________
28. When a valid argument is truth-tabled what is the nature of the last column of the truth-table computed?	________	________

	Did not Check Back	Checked Back
29. Under what conditions are the truth values in one column of a truth-table all the same?		
30. Construct a truth-table for the propositional function. $[p \supset (q\ .\ r)]$		
31. If the function $(p \supset q) \equiv (\sim q \supset \sim p)$ is logically true, is the function $\sim[(p \supset q) \equiv (\sim q \supset \sim p)]$ logically false? Why?		
32. Since every contingently true expression is said to be logically equivalent to every other contingently true expression, does this mean that there are no differences between contingently true statements?		

	Did not Check Back	Checked Back
33. What conclusion follows from the following two premises? A is taller than B C is shorter than B		
34. Put the following argument into symbolic form. If the weather is favorable, we will work outside. The weather is favorable, therefore, we will work outside.		
35. What is the propositional function in 34 for which a truth-table can be constructed? (Conjoin the premises and indicate that the premises imply (⊃) the conclusion.)		
36. Construct the truth-table for the propositional function you wrote in 35.		

III. Answer items 37-45 by doing any review of the instructional materials that is necessary.

	Did not Check Back	**Checked Back**
37. Put the following argument into symbolic form. **If the weather is favorable, we will work outside. We do not, work outside. Therefore, the weather is not favorable.**		
38. What is the propositional function in 37 for which a truth-table can be constructed?		
39. Construct the truth-table for the propositional function you wrote in 38.		
40. Put the following argument into symbolic form. **If the weather is favorable, we will work outside. The weather is not favorable. Therefore we will not work outside.**		

	Did not Check Back	Checked Back
41. What is the propositional function in 40 for which a truth-table can be constructed?		
42. Construct the truth-table for the propositional function you wrote in 41.		
43. Put the following argument into symbolic form. If Jones wins the election, then we will either have a depression or war. But it is clear that there can be no war. Therefore, it is impossible for Jones to win the election.		
44. What is the propositional function in 43 for which a truth-table can be constructed?		
45. Construct the truth-table for the propositional function you wrote in 44.		

SYMBOLIC LOGIC

PROPOSITIONAL CALCULUS - TRUTH-TABLE CONSTRUCTION

AND ANALYSIS

Mode of Presentation II

The subject matter of formal logic is the totality of organized systematized, and related argument forms. Symbolic logic is formal logic with more emphasis placed on the development of an adequate symbolism.

The propositional calculus - also known as the sentential calculus - is a basic part of logic and is the starting point for the study of symbolic logic.

A "proposition" is any sentence which must be either true or false but which cannot be both true and false. "a" is a proposition, "b" is not.

(a) The cat is on the mat.

(b) The pain in my arm is blue.

The calculus of propositions is not concerned with the subject-matter of propositions, but only with the essential feature common to all propositions - their capacity for truth or falsehood.

The truth or falsity of a proposition is its truth-value.

The basic material of the propositional calculus is of two kinds:

(1) Letters of the alphabet p, q, r, etc., which stand for propositions. These letters, as used in the propositional calculus, are known as propositional variables because they stand indifferently for any proposition.

(2) Words like "not", "and", "or", and "if ---then---" which link the propositional variables into truth-functions; i.e., compound expressions like "not p or q" or "if p, then q" whose truth-values depend on the truth-values of their component propositions, are known as logical constants.

The logical constants are represented by symbols, thus:

For "not p", write "~p".

For "p and q", write "p · q".

For "p or q", write "p v q".

For "if p, then q" write "p ⊃ q".

Since a proposition must be either true or false, and since when p is true "not p" is false, and conversely, the following matrix represents all possibilities.

p	not p
true	false
false	true

When two variables are considered, the matrix must contain two columns and four rows.

p	q
true	true
true	false
false	true
false	false

That is, combinations of p and q must contain all possible permutations of "true" and "false".

If negations are added, the matrix contains four columns as well as four rows:

p	~p	q	~q
true	false	true	false
true	false	false	true
false	true	true	false
false	true	false	true

The conjunctive function

From the given truth or falsity of the propositional variables, p and q, the truth function of the compound proposition (p and q), symbolized as $(p \cdot q)$ is determined and represented in the following matrix, known as a truth-table.

p	q	$(p \cdot q)$
true	true	true
true	false	false
false	true	false
false	false	false

The conjunctive function, as a unit, can only be true when both components are true.

Consider the example, "if it is both Friday and the thirteenth, then you will have bad luck". For the antecedent to be true, i.e. for you to expect bad luck, the day must be both Friday and the thirteenth.

The Disjunctive Function

The disjunctive function (p or q), symbolized $(p \vee q)$, is true when either or both of the components are true. The truth-table is represented below:

p	q	$(p \vee q)$
true	true	true
true	false	true
false	true	true
false	false	false

Thus, if (p v q), as a unit, is the antecedent, the truth of either is sufficient for the truth of the consequent.

If membership in a major political party is a requirement for public office, being either a Democrat or a Republican would be sufficient, but presumably a candidate could not be both.

On the other hand, a person may qualify for voting rights in a district if he either resides or owns property there. He may meet both requirements, but in either case the disjunctive function, either a resident or a property owner, would be true.

Since when p is true, ~p is false, a truth table can be given for (~ p v q)

p	~p	q	(~ p v q)
true	false	true	true
true	false	false	false
false	true	true	true
false	true	false	true

A verbal instance of (~ p v q) is "Either he is not a candidate or he will attend the convention."

This statement is equivalent to "If he is a candidate, then he will attend the convention."

The Implication Function

The meaning of "if p, then q" is substantially the same as (~ p v q). Therefore the implicative function (p ⊃ q) is said to be equivalent to (~ p v q). The truth table for the implicative function is hence the same as the truth tables for (~p v q).

p	q	$(p \supset q)$
true	true	true
true	false	false
false	true	true
false	false	true

Thus, the implicative function is true when a false antecedent yields a false consequent. It is false when a true antecedent yields a false consequent. And it is true when a false antecedent yields a true consequent.

It may seem absurd to state that the function "If Caesar killed Brutus, then $7 + 5 = 12$" is true, but this interpretation is merely broader than idiomatic English, it is not different.

An example of an idiomatic expression is, "If he passes that examination, I will eat my hat". This assertion is intended to be true, even though both the antecedent and consequent are considered false. It is thus an example of the last row in the truth table. It could easily be changed to, "If he passes that examination, I will eat my lunch," when the consequent is true; i.e., an example of the third row of the truth table.

This implication may be trivially true, but this does not matter; it is not false.

Primitive Signs and Defined Signs

Just as it is possible to define "$\supset$" in terms of " v " so it is possible to define both "v" and " . " in terms of other logical constants. But "$\sim$" has to be taken as a primitive idea which is undefinable in terms of the other constants.

For the purposes of the propositional calculus, the truth table of each of the propositional functions is taken as a definition of that function and as a rule for the use of the logical constants occurring in the function.

For this reason, any functions having the same truth-table are equivalent and interchangeable for logical purposes.

The truth-table for the expression $\sim(p \cdot \sim q)$ is derived and presented below:

p	q	$\sim q$	$(p \cdot \sim q)$	$\sim(p \cdot \sim q)$
true	true	false	false	true
true	false	true	true	false
false	true	false	false	true
false	false	true	false	true

The truth table for $\sim(p \cdot \sim q)$ is the same as the truth table for $(p \supset q)$ therefore, the expressions $\sim(p \cdot \sim q)$ and $(p \supset q)$ are equivalent.

The Equivalence Function

The equivalence function "$\equiv$" is true if both members of the equivalence are true or if both members are false.

p	q	$(p \equiv q)$
true	true	true
true	false	false
false	true	false
false	false	true

The equivalence function can also be defined in terms of the implicative function.

p	q	$(p \supset q)$	$(q \supset p)$	$(p \supset q) \cdot (q \supset p)$
true	true	true	true	true
true	false	false	true	false
false	true	true	false	false
false	false	true	true	true

The truth tables for $p \equiv q$ and $(p \supset q) \cdot (q \supset p)$ are the same, therefore the two expressions are logically equivalent.

For example, the statement, "If February has 29 days, then it is leap year" can be reversed and still be true. Hence the equivalence symbol could be appropriately used.

Another way to state the equivalence function verbally is to say "p if and only if q".

Scope of Logical Constants

The scope of a logical constant is defined as those parts of the truth-functional expression which are controlled by the constant.

The constant " ~ " controls or operates over the expression (or part of the expression) which follows it.

The constants " · " , " v ", and " ⊃ " control or operate over the expression (or part of the expression) both preceding and following them.

These are called binary constants.

The scope of "~ " is confined to the propositional variable immediately succeeding it, except where "~ " is followed by a left-handed bracket " (". In this case the scope of " ~ " extends to the following bracket.

Example 1

In $\sim p . q$, the scope of "$\sim$" is p.

Example 2

In $\sim [p \cdot ($ [illegible] $\supset r) \vee s]$, the scope of "$\sim$" is the whole expression.

Example 3

In $\sim (p \cdot q) \vee (r \supset s)$, the scope of "$\sim$" is $(p \cdot q)$.

If a binary constant is flanked by a propositional variable, the scope of the constant on that side is confined to the propositional variable. If a binary constant is flanked by a bracket, the scope of the constant extends to the corresponding bracket.

Example 1

In the expression $p \supset (q \vee r)$, the scope of "$\supset$" is the whole expression i.e., p and $(q \vee r)$.

Example 2

In the expression

$$[(p \supset q) \supset r\} \supset [(r \supset p) \supset (s \supset p)]$$

the scope of the first "$\supset$" is p and q the scope of the second "$\supset$" is $(p \supset q)$ and r the scope of the third "$\supset$" is the whole expression, etc.

The Classification of Propositions

Any truth-functional expression may be truth-tabled and classified into one of three types in accordance with the character of its truth-table. If the truth-table contains only T's, the expression is valid or logically true or a tautology. If the truth table contains only F's, then the expression is logically false or contradictory. If the truth table contains both T's and F's (as is the case most of the time), then the expression is contingent. In this case the truth or falsity depends upon a particular combination of truth-values of the variables, being true for some possibilities and false for others.

For example, the equivalence function, $(p \supset q) \equiv (\sim q \supset \sim p)$ can be classified by truth-table analysis. (The number headings indicate the order of computation.)

(1)	(2)	(3)	(4)	(5)
p	q	$\sim p$	$\sim q$	$p \supset q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

(5)	(7)	(6)
$(p \supset q)$	$\equiv$	$(\sim q \supset \sim p)$
T	T	T
F	T	F
T	T	T
T	T	T

Since the last column (7) computed contains all T's, the truth-function in question is logically true.

The conjunctive function (p ⊃ q) · (p · ∼q) can also be classified by truth-table analysis.

(1) p	(2) q	(3) ∼q	(4) (p ⊃ q)	(5) (p · ∼q)
T	T	F	T	F
T	F	T	F	T
F	T	F	T	F
F	F	T	T	F

(4) (p ⊃ q)	(6) ·	(5) (p · ∼q)
T	F	F
F	F	T
T	F	F
T	F	F

Since column (6) contains all F's, the truth-function is therefore contradictory.

The classification of propositions yields two important rules:

(1) Every logically false or contradictory expression is the negation of a logically true or tautologous expression, and vice versa.

(2) Every logically true expression is logically equivalent to every other logically true expression; and similarly every contingent expression is logically equivalent to every other contingent statement.

Truth-Tables as a Decision Procedure

Truth-tables provide an automatic method of deciding in the case of an expression in the propositional calculus whether or not it is logically true or tautological.

If the expression is in the form of an argument, truth-tables provide a means of deciding whether any argument which falls within the scope of the propositional calculus is valid or not.

Truth tables provide a meansfor determining the validity of arguments independently of the truth-property. If a premise is assumed to be true, the truth property will be preserved in a valid argument.

Construction and Application of Truth Tables

First construct the columns of truth-tables for the individual propositional variables, beginning at the left and moving toward the right. Then construct the columns for the constants of narrowest scope.

For example the truth-table for the propositional function

$$[(p \supset q) \cdot \sim q] \supset \sim p$$

would be constructed as follows:

(1)	(2)	(3)	(4)	(5)	(6)
p	q	$\sim p$	$\sim q$	$(p \supset q)$	$[(p \supset q) \cdot \sim q$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	F	T	F
F	F	T	T	T	T

(6)	(7)	(3)
$[(p \supset q) \cdot \sim q$	$\supset$	$\sim p$
F	T	F
F	T	F
F	T	T
T	T	T

Since column (7), the last computed, contains only T's, the propositional function is a tautology.

Illustrative Examples

The following arguments can be checked for validity using truth-tables.

(1) If A is elected, then B will resign. If C is elected, then B will not resign. Therefore, if A is elected, then C will not be elected.

The premises are symbolized as follows:

$$(A \supset B), \quad (C \supset \sim B)$$

The conclusion is $(A \supset \sim C)$

The entire argument is therefore represented in symbolic form as:

$$[(A \supset B) \cdot (C \sim B)] \supset (A \supset \sim C)$$

The validity of the argument is then tested by truth-table analysis.

(1) A	(2) B	(3) C	(4) $\sim B$	(5) $\sim C$	(6) $(A \supset B)$	(7) $(C \supset \sim B)$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	T	F	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

(6) $[(A \supset B)$	(9) $.$	(7) $(C \supset \sim B)]$	(10) $\supset$	(8) $(A \supset \sim C)$
T	F	F	T	F
T	T	T	T	T
F	F	T	T	F
F	F	T	T	T
T	F	F	T	T
T	T	T	T	T
T	T	T	T	T
T	T	T	T	T

Since column (10), the last column computed, contains only T's, the argument is valid.

(2) If there is an election, the government will not remain in power. Either the government will remain in power or there will be a coup d' etat, there won't be a coup d' etat. Therefore, there won't be an election.

Letting the letters, E, P, and C represent "election", "power", and "coup" respectively, the premises of the argument become:

$(E \supset \sim P)$, $(P \vee C)$, and $\sim C$

The conclusion is $\sim E$.

The entire argument in symbolic form is thus:

$$\left([(E \supset \sim P) \;.\; (P \vee C)] \;. \sim C\right) \; C \; \sim E$$

The validity of this propositional function is then tested by truth-table analysis:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
E	$\sim E$	P	$\sim P$	C	$\sim C$	$(E \supset \sim P)$	$(P \vee C)$
T	F	T	F	T	F	F	T
T	F	T	F	F	T	F	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	F	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F

(7)	(9)	(8)	(10)	(6)	(11)	(2)
$([(E \supset \sim P)$	$.$	$(P \vee C)]$	$.$	$\sim C)$	$\supset$	$\sim E$
F	F	T	F	F	T	F
F	F	T	F	T	T	F
T	T	T	F	F	T	F
T	F	F	F	T	T	F
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	F	F	F	T	T	T

The final column contains all T's. Therefore the argument is valid.

(3) If Jones is guilty or the police are suspicious, then either Jones will bribe the police or he will remain in town. Therefore, if Jones remains in town or the police are suspicious, it is false that he is both guilty and is not bribing the police.

Using the following symbols, G, S, B, and R to represent "guilty", "suspicious", "bribe", and "remain", respectively, the premises become:

$$(G \vee S) \supset (B \vee R)$$

The conclusion is: $(R \vee S) \supset \sim (G \,.\, \sim B)$

The entire argument is thus: $[(G \vee S) \supset (B \vee R)] \supset [(R \vee S) \supset \sim (G \,.\, \sim B)]$
Since there are four propositional variables, the truth table contains sixteen rows.

(1) G	(2) S	(3) B	(4) ~B	(5) R	(6) ~R	(7) (G v S)	(8) (B v R)
T	T	T	F	T	F	T	T
T	T	T	F	F	T	T	T
T	T	F	T	T	F	T	T
T	T	F	T	F	T	T	F
T	F	T	F	T	F	T	T
T	F	T	F	F	T	T	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F
F	F	T	F	T	F	F	T
F	F	T	F	F	T	F	T
F	F	F	T	T	F	F	T
F	F	F	T	F	T	F	F

(7)	(13)	(8)	(14)	(9)	(12)	(11)	(10)
[(G v S)	⊃	(B v R)]	⊃	[(R v S)	⊃	~	(G . ~B)]
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	FT	T	F	T	F	F	T
T	F	F	T	T	F	F	T
T	T	T	T	T	T	T	F
T	T	T	T	F	T	T	F
T	T	T	F	T	F	F	T
T	F	F	T	F	T	F	T
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	F	F	T	T	T	T	F
F	T	T	T	T	T	T	F
F	T	T	T	F	T	T	F
F	T	T	T	T	T	T	F
F	T	T	T	F	T	T	F

The truth-table for the last constant (14th column) contains two "F's" Therefore, the argument is invalid.

STUDY GUIDE FOR PRESENTATION 'I'

Name ______________________________

Time required ______ ______ ________

Answer each question, or solve the problem, in the space provided.

Check the blank at the left if you know the answer, or can solve the problem, without checking back in the instructional materials.

Check the blank at the right if it is necessary to look up the answer or look up how to solve the problem.

Leave both blanks unchecked if you cannot respond at all.

I. Answer questions 1 - 21 after completing the equivalence function (MIDDLE OF PAGE 8)

	Did not Check Back	Checked Back
1. What is a proposition?	________	______
2. Are all assertions propositions?	________	______
3. Is the following a proposition? John is 16 years old.	________	______
4. In your own words, give a definition of 'logical constant'.	________	______
5. On what does the truth-value of the logical constants depend?	________	______

	Did not Check Back	Checked Back
6. Given the two propositions: (a) The sun is shining. (b) It is cloudy. Link these into a compound proposition using a logical constant		
7. What is the truth-table for (p . q)? (last column)		
8. Why is the conjunctive function false for all cases except one?		
9. What is the truth-value of John is big and strong if it is false that John is big and true that John is strong.		
10. What is the truth-table for (p v q)?		
11. Why is the disjunctive function false for only one case?		

	Did not check back	Checked Back
12. What is the truth-value of John is big or strong when it is false that John is big and true that John is strong?	________	________
13. What is the truth-table for $(p \supset q)$?	________	________
14. Why is the implicative function true when p and q are both true and also when p and q are both false?	________	________
15. Give an example of the third row of the truth-table for $(p \supset q)$?	________	________
16. When are two propositional functions considered logically equivalent?	________	________
17. The truth-tables for the two functions $(p \vee q)$ and $(q \vee p)$ are the same. Therefore these two functions are ______________	________	________

	Did not Check Back	Checked Back
18. Give two simple propositional functions which you would predict are logically equivalent?		
19. What is the truth-table for ($p \equiv q$)?		
20. Why is the equivalence function true when both members are false?		
21. Give a verbal instance of the equivalence function.		

II. Answer items 22 - 36 after completing Presentation

22. The logical constants 'v', '.', etc. are called binary constants because they operate on whatever is on both sides of them. What constant is not a binary constant?		
23. The logical constants need help in linking propositional variables. What symbols provide this function?		

	Did not Check Back	Checked Back
24. Put the following propositional function into symbolic form using the appropriate symbols. If John is sick, then either he will send a note or call by telephone.		
25. Truth tables provide a means of classifying arguments into three classes. What are these?		
26. Arguments in which the conclusion contradicts one or both of the premises and arguments in which the conclusion may be true part of the time are both called invalid. Why are both of these arguments invalid?		
27. Give an example of an argument when the conclusion logically follows from the premises.		
28. When a valid argument is truth-tabled what is the nature of the last column of the truth-table computed?		

	Did not Check Back	Checked Back
29. Under what conditions are the truth values in one column of a truth-table all the same?	________	________
30. Construct a truth-table for the propositional function. $[p \supset (q \,.\, r)]$	________	________
31. If the function $(p \supset q) \equiv (\sim q \supset \sim p)$ is logically true, is the function $\sim[(p \supset q) \equiv (\sim q \supset \sim p)]$ logically false? Why?	________	________
32. Since every contingently true expression is said to be logically equivalent to every other contingently true expression, does this mean that there are no differences between contingently true statements?	________	________

	Did not Check Back	Checked Back
33. What conclusion follows from the following two premises? A is taller than B C is shorter than B	________	________
34. Put the following argument into symbolic form. If the weather is favorable, we will work outside. The weather is favorable, therefore, we will work outside.	________	________
35. What is the propositional function in 34 for which a truth-table can be constructed? (Conjoin the premises and indicate that the premises imply (⊃) the conclusion.)	________	________
36. Construct the truth-table for the propositional function you wrote in 35.	________	________

	Did not Check Back	Checked Back
III. Answer items 37-45 by doing any review of the instructional materials that is necessary.		
37. Put the following argument into symbolic form. If the weather is favorable, we will work outside. We do not, work outside. Therefore, the weather is not favorable.	________	________
38. What is the propositional function in 37 for which a truth-table can be constructed?	________	________
39. Construct the truth-table for the propositional function you wrote in 38.	________	________
40. Put the following argument into symbolic form. If the weather is favorable, we will work outside. The weather is not favorable. Therefore we will not work outside.	________	________

	Did not Check Back	Checked Back
41. What is the propositional function in 40 for which a truth-table can be constructed?		
42. Construct the truth-table for the propositional function you wrote in 41.		
43. Put the following argument into symbolic form. If Jones wins the election, then we will either have a depression or war. But it is clear that there can be no war. Therefore, it is impossible for Jones to win the election.		
44. What is the propositional function in 43 for which a truth-table can be constructed?		
45. Construct the truth-table for the propositional function you wrote in 44.		

SYMBOLIC LOGIC

PROPOSITIONAL CALCULUS-TRUTH TABLE CONSTRUCTION AND ANALYSIS

Mode of Presentation III

INTRODUCTION

The subject-matter of formal logic is the totality of organized, systematized, and related argument forms. Symbolic logic is formal logic with more emphasis placed on the development of an adequate symbolism.

The propositional calculus - also known as the sentential calculus - is a basic part of logic and is the starting point for the study of symbolic logic.

DEFINITIONS

A 'proposition' is any sentence which must be either true or false but which cannot be both true and false: 'a' is a proposition but 'b' and 'c' are not.

(a) This pencil is blue.

(b) This pain is blue.

(c) John is both present and absent.

The calculus of propositions is not concerned with the subject-matter of propositions, but only with the essential feature common to all propositions - their capacity for truth or falsehood.

The truth or falsity of a proposition is its truth-value.

The basic components of the propositional calculus are of two kinds:

(1) Lower case letters of the alphabet (p, q, r, etc.) which stand for propositions. These letters, as used in the propositional calculus, are known as propositional variables because they stand indifferently for any proposition.

(2) Words like 'not', 'and', 'or', and 'if -- then --' which link the propositional variables into truth-function; i.e., compound expressionslike 'not p or q' or 'if p, then q' whose truth-values depend on the truth values of their component propositions, are known as logical constants.

The logical constants are represented by symbols, thus:

For 'not p', write '$\sim p$'
For 'p and q', write '$p \, . \, q$'
For 'p or q', write '$p \vee q$'
For 'if p, then q', write '$p \supset q$'

For example, if 'brown' is symbolized by p and 'blue' is symbolized by q, the propositional function 'not brown or not blue' would be written thus:

$$\sim p \vee \sim q$$

Similarly, if 'brown' is symbolized by p and 'strong' is symbolized by q, 'if brown then strong' is written as:

$$p \supset q$$

TRUTH TABLES

Since a proposition must be either true or false, and since when p is true, not p is false, and conversely, the following matrix (known as a truth-table) represents all possibilities.

p	not p
true	false
false	true

When two variables are considered, the matrix must contain two columns and four rows.

p	q
true	true
true	false
false	true
false	false

That is, combinations of p and q must contain all possible permutations of 'true' and 'false'.

If negations are added, the matrix contains four columns as well as four rows:

p	~ p	q	~ q
true	false	true	false
true	false	false	true
false	true	true	false
false	true	false	true

THE CONJUNCTIVE FUNCTION

From the given truth or falsity of the propositional variables, p and q, the truth function of the compound proposition, p and q, (symbolized as p . q) is determined and represented in the following truth-table.

p	q	(p . q)
true	true	true
true	false	false
false	true	false
false	false	false

An example of the conjunctive function is:

If it is Friday and the weather is clear,
then I will go to Chicago.

i.e., if (p . q) then r.

Thus, if either p or q is false, then the conditions for r are not met. Clearly, if both p and q are false, the conditions are not met.

THE DISJUNCTIVE FUNCTION

The disjunctive function (p or q) symbolized (p v q) is true, when either or both of the components are true. The truth-table is represented below:

p	q	(p v q)
true	true	true
true	false	true
false	true	true
false	false	false

Thus, if (p v q), as a unit, is the antecedent, the truth of either is sufficient for the truth of the consequent.

If membership in a major political party is a requirement for public office, being either a Democrat or a Republican would be a sufficient condition for public office. However, presumably a candidate could not be both.

On the other hand, a person may qualify for voting rights in a district if he either resides or owns property there; he may meet both requirements, but in either case the disjunctive function 'either a resident or a property owner' would be true.

Since when p is true, $\sim p$ is false, a truth-table can be given for $(\sim p \vee q)$

p	$\sim p$	q	$(\sim p \vee q)$
true	false	true	true
true	false	false	false
false	true	true	true
false	true	false	true

Note that $(\sim p \vee q)$ is false only when both of the components are false.

A verbal instance of $(\sim p \vee q)$ is 'Either he is not a candidate or he will attend the convention.' This statement is equivalent to, 'If he is a candidate, then he will attend the convention.'

Another instance of $(\sim p \vee q)$ is, 'Either he will not pass the exam or he will be promoted.' The equivalent form is, 'If he passes the exam, then he will be promoted.'

THE IMPLICATIVE FUNCTION

Since instances of $(\sim p \vee q)$ and $(p \supset q)$ are semantically equivalent, the two functions are defined as equivalent for logical purposes. The truth-tables for the two functions are, hence, the same.

p	q	$(p \supset q)$
true	true	true
true	false	false
false	true	true
false	false	true

Thus, the implicative function is true when a true antecedent yields a true consequent. It is false when a true antecedent yields a false consequent. It is true when a false antecedent yields a true consequent and it is also true when both the antecedent and consequent are false.

It may seem absurd to state that the function, 'If Caesar killed Brutus, the 7 + 5 = 12', is true, but this interpretation is merely broader than idiomatic English; it is not different.

An example of an idiomatic expression is, 'If he passes that examination, I will eat my hat.' This assertion is intended to be true, even though both the antecedent and consequent are considered false. It is thus an example of the last row in the truth-table. It could easily be changed to, "If he passes that examination, I will eat my lunch', where the consequent is true; i.e., this would be another example of the third row in the truth-table. The implication may be trivally true, but this does not matter, it is not false.

PRIMITIVE SIGNS AND DEFINED SIGNS

Just as it is possible to define '⊃', in terms of 'v', so it is possible to define both 'v' and '.' in terms of other logical constants. But '~' has to be taken as a primitive idea which is undefinable in terms of the other constants.

For the purposes of the propositional calculus, the truth-table of each of the propositional functions is taken as a definition of that function and as a rule for the use of the logical constant occurring in the function.

For this reason, any functions having the same truth-table are equivalent and interchangeable for logical purposes.

The truth-table for the expression $\sim (p \,.\, \sim q)$ is derived and presented below:

p	q	$\sim q$	$(p \,.\, \sim q)$	$\sim (p. \sim q)$
true	true	false	false	true
true	false	true	true	false
false	true	false	false	true
false	false	true	false	true

The truth-table for $\sim (p \,.\, \sim q)$ is the same as the truth-table for $(p \supset q)$. Therefore the expressions $\sim (p \,.\, \sim q)$ and $(p \supset q)$ are defined as logically equivalent.

For example, the statement 'If it is Friday, I will go to New York' is logically equivalent to ' It is not the case that it is both Friday and I do not go to New York'.

THE EQUIVALENCE FUNCTION

Note that when the propositional variables in the function $(p \supset q)$ are reversed to form $(q \supset p)$ the second and third rows in the truth-table are reversed.

p	q	$(p \supset q)$	$(q \supset p)$
true	true	true	true
true	false	false	true
false	true	true	false
false	false	true	true

If the two functions $(p \supset q)$ and $(q \supset p)$ are conjoined with the connective (.), the following truth-table is formed. (Remembering that '.' is false if either member is false.)

$(p \supset q)$	$(q \supset p)$	$(p \supset q) \, . \, (q \supset p)$
true	true	true
false	true	false
true	false	false
true	true	true

The function $(p \supset q) \, . \, (q \supset p)$ can also be written as $p \equiv q$, or p is equivalent to q. The corresponding truth-table is thus that of the equivalence function. Stated verbally, the function becomes, 'p is a necessary and a sufficient condition for q or conversely' An instance of $(p \equiv q)$ is 'If February has 29 days, then it is leap year and if it is leap year, then February has 29 days'.

Still another form of the equivalence function is 'p if and only if q'.

SCOPE OF LOGICAL CONSTANTS

The scope of a logical constant is defined as those parts of the truth-functional expression which are controlled by the constant. The constant '~' controls or operates over the expression (or part of the expression) both preceding and following them. These are called binary constants.

The scope of '~' is confined to the propositional variable immediately succeeding it except where '~' is followed by a left-handed bracket, '('. In this case the scope of '~' extends to the following bracket.

Example 1

In $\sim p$, the scope of '$\sim$' is p.

Example 2

In $\sim [p.(q \supset r) \vee s]$, the scope of '$\sim$' is the entire expression.

Example 3

In $\sim(p \,.\, q) \vee (r \supset s)$, the scope of '$\sim$' is $(p \,.\, q)$

If a binary constant is flanked by a propositional variable, the scope of the constant on that side is confined to the propositional variable. If a binary constant is flanked by a bracket, the scope of the constant extends to the corresponding bracket.

Example 1

In the expression $p \supset (q \vee r)$, the scope of '$\supset$' is the whole expression; i.e., p and $(q \vee r)$

Example 2

In the expression $[(p \supset q) \supset r] \supset [(r \supset p) \supset (s \supset p)]$
the scope of the first '$\supset$' is p and q;
the scope of the second '$\supset$' is $(p \supset q)$ and r;
the scope of the third '$\supset$' is the whole expression, etc.

THE CLASSIFICATION OF PROPOSITIONS

Any truth-functional expression may be truth-tabled and classified into one of three types in accordance with the character of its truth-table. If the truth-table contains only T's, the expression is valid or logically true or a tautology. If the truth-table contains only F's, the expression is logically false or contradictory. If the truth-table contains both T's and F's, (as is the case most of the time), then the expression is contingent. In this case the truth or falsity depends upon a particular combination of truth-values of the variables -- being true for some possibilities and false for others. For example, the conclusion 'Some men are brave' does not validly follow from the premise 'Some men fight in wars' and 'Some men who fight in wars are brave'. The conclusion would probably be true for some group of men and false for others. But, since the quantifier 'some' is used, there is no way to tell whether the conclusion is true or false. This fact is represented by the truth-table containing both T's and F's.

For example the equivalence function $(p \supset q) \equiv (\sim q \supset \sim p)$ can be classified by truth-table analysis. (The numeral headings indicate the order of computation.)

(1)	(2)	(3)	(4)	(5)
p	q	$\sim p$	$\sim q$	$(p \supset q)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

(5)	(7)	(6)
$(p \supset q)$	$\equiv$	$(\sim q \supset \sim p)$
T	T	T
F	T	F
T	T	T
T	T	T

Since the last column computed (7) contains all T's, the truth function in question is logically true.

The conjunctive function (p ⊃ q) . (p . ~ q) can also be classified by truth-table analysis.

(1)	(2)	(3)	(4)	(5)
p	q	~ q	(p ⊃ q)	(p . ~ q)
T	T	F	T	F
T	F	T	F	T
F	T	F	T	F
F	F	T	T	F

(4)	(6)	(5)
(p ⊃ q)	.	(p . ~ q)
T	F	F
F	F	T
T	F	F
T	F	F

Since column (6), the last constructed, contains only F's the truth-function (p ⊃ q) . (p . ~ q) is therefore logically false, or self-contradictory. This can also be understood by noting that when (p ⊃ q) and p are given, then q must follow. For example, if someone says 'If it rains, the streets will be wet' and then asserts 'It is raining' and 'The streets are not wet', he has contradicted himself.

The classification of propositions yields two important rules:

(1) Every logically false or contradictory expression is the negation of a logically true or tautologous expression and vice-versa.

(2) Every logically true expression is logically equivalent to every other logically true expression; and similarly every contingent expression is logically equivalent to every other contingent statement.

Construction and Application of Truth Tables

When constructing truth-tables, begin at the left and move toward the right. Start with the individual variables, and proceed next to the constants of narrowest scope. Numbered headings may be used to indicate order of computation. Whenever it is necessary, for convenience, to rewrite a column, the same numbered heading should be used.

For example, the truth table for the propositional function:

$$[(p \supset q) \cdot \sim q] \supset \sim p$$

could be constructed as follows:

(1) p	(2) q	(3) $\sim p$	(4) $\sim q$	(5) $(p \supset q)$	(6) $(p \supset q) \cdot \sim q$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	F	T	F
F	F	T	T	T	T

(6) $[(p \supset q) \cdot \sim q]$	(7) $\supset$	(3) $\sim p$
F	T	F
F	T	F
F	T	T
T	T	T

Column (7) contains only T's. Hence, the propositional function in question is logically true or a tautology. Note that columns (6) and (3) were recopied for ease in computing column (7).

Only arguments as propositional functions involving two variables have been discussed. Most arguments, of course, contain more than two variables. Consider the following:

(1) If it rains, the streets will be slick.
(2) If the streets are slick, there will be accidents.
(3) Therefore, if it rains, there will be accidents.

Letting the letters R, S, and A represent "rains", "slick", and "accidents", respectively the argument is reduced to the form:

$$[(R \supset S) \cdot (S \supset A)] \supset (R \supset A)$$

The truth table is constructed as follows:

(1)	(2)	(3)	(4)	(6)	(5)	(8)	(7)
R	S	A	$[(R \supset S)$	$\cdot$	$(S \supset A)]$	$\supset$	$(R \supset A$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Column (8) contains all T's, therefore, the argument is valid.

Note that when an argument involves three propositional variables the truth table must contain eight rows in order to insure that all possible permutations of T's and F's will be included for any two given variables.

Illustrative Examples

1. The following arguments may be checked for validity using truth tables:

If A is elected then B will resign.
If C is elected, then B will not resign.
Therefore, if A is elected, then C will not be elected.

The premises are symbolized as follows:

$$(A \supset B), (C \supset \sim B)$$

The conclusion is the following:

$$(A \supset \sim C)$$

The entire argument is therefore represented as the following propositional function:

$$[(A \supset B) \;.\; (C \supset \sim B)] \supset (A \supset \sim C)$$

The truth-table is constructed as follows:

(1) A	(2) B	(3) C	(4) $\sim B$	(5) $\sim C$	(6) $(A \supset B)$	(7) $(C \supset B)$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	T	F	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

(6) $[(A \supset B)$	(9) $.$	(7) $(C \supset \sim B)]$	(10) $\supset$	(8) $(A \supset \sim C)$
T	F	F	T	F
T	T	T	T	T
F	F	T	T	F
F	F	T	T	T
T	F	F	T	T
T	T	T	T	T
T	T	T	T	T
T	T	T	T	T

Since the last column computed (10) contains only T's the argument is valid.

2. If there is an election, the government will not remain in power. Either the government will remain in power or there will be a coup d' etat. There will not be a coup d' etat. Therefore, there will not be an election.

Letting the letters E, P, and C represent "election", "power", and "coup", respectively, the premises of the argument become:

$$[E \supset \sim P), \quad (P \vee C), \text{ and } \sim C$$

The entire argument in symbolic form is thus:

$$\Big([(E \supset \sim P) \,.\, (P \vee C)] \,.\, \sim C\Big) \supset \sim E$$

The validity of the argument is then tested by truth-table analysis:

(1) E	(2) ~E	(3) P	(4) ~P	(5) C	(6) ~C	(7) (E⊃~P)	(8) (P v C)
T	F	T	F	T	F	F	T
T	F	T	F	F	T	F	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F

(7) ([(E⊃~P)	(9) .	(8) (P v C)]	(10) .	(6) ~C))	(11) ⊃	(2) ~E
F	F	T	F	F	T	F
F	F	T	F	T	T	F
T	T	T	F	F	T	F
T	F	F	F	T	T	F
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	F	F	F	T	T	T

The final column contains all T's. Therefore the argument is valid.

3. If there is a change in government, then confidence in the dollar will not be restored. If confidence in the dollar is not restored, then imports will be restricted. Therefore, if imports are not restricted, there will not be a change in government.

Let the symbols "Ch", "Co", and "R" represent "change", "confidence", and "restored" respectively. Then the premise becomes:

$$(Ch \supset {\sim}Co) \text{ and } ({\sim}Co \supset R)$$

The conclusion is:

$$({\sim}R \supset {\sim}Ch)$$

The truth-table is constructed as follows:

(1) Ch	(2) ~Ch	(3) Co	(4) ~Co	(5) R	(6) ~R	(7) $(Ch \supset {\sim}Co)$	(9) .	(8) $({\sim}Co \supset R)$
T	F	T	F	T	F	F	T	F
T	F	T	F	F	T	F	F	T
T	F	F	T	T	F	T	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	T	F	T	T	T
F	T	T	F	F	T	T	T	T
F	T	F	T	T	F	T	T	T
F	T	F	T	F	T	T	F	F

(7) $[(Ch \supset {\sim}Co)$	(9) .	(8) $({\sim}Co \supset R)]$	(11) $\supset$	(10) $({\sim}R \supset {\sim}Ch)$
F	F	F	T	T
F	F	T	T	F
T	T	T	T	T
T	F	F	T	F
T	T	T	T	T
T	T	T	T	T
T	T	T	T	T
T	F	F	T	T

Column (11) computed from column (9) and column (10), contains only T's. Therefore the argument is valid.

4. If Jones resigns, the party will split and there will be an election. If the party splits and there is an election, the international situation will not improve. But the international situation does improve. Therefore, Jones won't resign.

Letting the letters R, S, E, and I represent "resign, "split", "election" and "international situation" respectively, the premises become:

$$R \supset (S . E) \quad , \quad (S . E) \supset \sim I \ \text{ and } I$$

and the conclusion becomes:

$$\sim R$$

The entire argument, in symbolic form is thus:

$$\left([R \supset (S . E)] \ . \ [(S . E) \supset \sim I] \ . \ I \right) \supset \sim R$$

The truth-table is constructed as follows: (note: that since there are four separate propositional variables, there must be sixteen rows in the truth table.)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
R	$\sim R$	S	E	I	$\sim I$	$(S . E)$	$[R \supset (S . E)]$
T	F	T	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	T	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F
T	F	F	T	F	T	F	F
T"	F	F	F	T	F	F	F
T	F	F	F	F	T	F	F
F	F	T	T	T	F	T	T
F	T	T	T	F	T	T	T
F	T	T	F	T	F	T	T
F	T	T	F	F	T	F	T
F	T	F	T	T	F	F	T
F	T	F	T	F	T	F	T
F	T	F	F	T	F	F	T
F	T	F	F	F	T	F	T

(8)	(10)	(9)	(11)	(5)	(12)	(2)
([R⊃(S . E)]	.	[(S . E)⊃~I]	.	I)	⊃	~R
T	F	F	F	T	T	F
T	T	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
T	F	F	F	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T

Column (12) contains only T's. Therefore the argument is valid.

5. If Jones is guilty or the police are suspicious, then either Jones will bribe the police or he will remain in town. Therefore, if Jones remains in town or the police are suspicious, it is false that he is both guilty and is not bribing the police.

Using the symbols G, S, B, and R, to represent "guilty", "suspicious", "bribe", and "remain", respectively, the premise becomes:

$$(G \lor S) \supset (B \lor R)$$

The conclusion is as follows:

$$(R \lor S) \supset \sim(G \,.\, \sim B)$$

The entire argument is thus:

$$[(G \lor S) \supset (B \lor R)] \supset [(R \lor S) \supset \sim(G \,.\, \sim B)]$$

Since there are four propositional variables, the truth-table contains sixteen rows.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
G	S	B	~B	R	~R	(G v S)	(B v R)
T	T	T	F	T	F	T	T
T	T	T	F	F	T	T	T
T	T	F	T	T	F	T	T
T	T	F	T	F	T	T	F
T	F	T	F	T	F	T	T
T	F	T	F	F	T	T	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F
F	F	T	F	T	F	F	T
F	F	T	F	F	T	F	T
F	F	F	T	T	F	F	T
F	F	F	T	F	T	F	F

(7)	(9)	(8)	(14)	(10)	(13)	(12)	(11)
[(G v S)	⊃	(B v R)]	⊃	[(R v S)	⊃	~	(G . ~B)]
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	T	F	F	T	F	F	T
T	F	T	T	T	F	F	T
T	T	T	T	T	T	T	F
T	T	T	T	F	T	T	F
T	T	F	F	T	F	F	T
T	F	T	T	F	T	F	T
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	T	F	T	T	T	T	F
T	F	T	T	T	T	T	F
F	T	T	T	T	T	T	F
F	T	T	T	F	T	T	F
F	T	F	T	T	T	T	F
F	T	T	T	F	T	T	F

The truth-table for the last column contains two F's, Therefore the argument is invalid.

STUDY GUIDE FOR PRESENTATION IV

Name ______________________________

Time required ______ ______ ______

Answer each question, or solve the problem, in the space provided.

Check the blank at the left if you know the answer, or can solve the problem, without checking back in the instructional materials.

Check the blank at the right if it is necessary to look up the answer or look up how to solve the problem.

Leave both blanks unchecked if you cannot respond at all.

I. Answer questions 1 - 21 after completing the equivalence function (end of page 12) .

	Did not Check Back	Checked Back
1. What is a proposition?	______	______
2. Are all assertions propositions?	______	______
3. Is the following a proposition? John is 16 years old.	______	______
4. In your own words, give a definition of 'logical constant'.	______	______
5. On what does the truth-value of the logical constants depend?	______	______

	Did not Check Back	Checked Back
6. Given the two propositions: (a) The sun is shining. (b) It is cloudy. Link these into a compound proposition using a logical constant	________	________
7. What is the truth-table for (p . q)? (last column)	________	________
8. Why is the conjunctive function false for all cases except one?	________	________
9. What is the truth-value of John is big and strong if it is false that John is big and true that John is strong.	________	________
10. What is the truth-table for (p v q)?	________	________
11. Why is the disjunctive function false for only one case?	________	________

	Did not check back	Checked Back
12. What is the truth-value of John is big or strong when it is false that John is big and true that John is strong?		
13. What is the truth-table for $(p \supset q)$?		
14. Why is the implicative function true when p and q are both true and also when p and q are both false?		
15. Give an example of the third row of the truth-table for $(p \supset q)$?		
16. When are two propositional functions considered logically equivalent?		
17. The truth-tables for the two functions $(p \vee q)$ and $(q \vee p)$ are the same. Therefore these two functions are ________________		

	Did not Check Back	Checked Back
18. Give two simple propositional functions which you would predict are logically equivalent?	________	________
19. What is the truth-table for $(p \equiv q)$?	________	________
20. Why is the equivalence function true when both members are false?	________	________
21. Give a verbal instance of the equivalence function.	________	________

II. Answer items 22 - 36 after completing Presentation IV

22. The logical constants 'v', '.', etc. are called binary constants because they operate on whatever is on both sides of them. What constant is not a binary constant?	________	________
23. The logical constants need help in linking propositional variables. What symbols provide this function?	________	________

	Did not Check Back	Checked Back
24. Put the following propositional function into symbolic form using the appropriate symbols. If John is sick, then either he will send a note or call by telephone.	________	________
25. Truth tables provide a means of classifying arguments into three classes. What are these?	________	________
26. Arguments in which the conclusion contradicts one or both of the premises and arguments in which the conclusion may be true part of the time are both called invalid. Why are both of these arguments invalid?	________	________
27. Give an example of an argument when the conclusion logically follows from the premises.	________	________
28. When a valid argument is truth-tabled what is the nature of the last column of the truth-table computed?	________	________

	Did not Check Back	Checked Back
29. Under what conditions are the truth values in one column of a truth-table all the same?		
30. Construct a truth-table for the propositional function. $[p \supset (q \, . \, r)]$		
31. If the function $(p \supset q) \equiv (\sim q \supset \sim p)$ is logically true, is the function $\sim [(p \supset q) \equiv (\sim q \supset \sim p)]$ logically false? Why?		
32. Since every contingently true expression is said to be logically equivalent to every other contingently true expression, does this mean that there are no differences between contingently true statements?		

	Did not Check Back	Checked Back
33. What conclusion follows from the following two premises? A is taller than B C is shorter than B		
34. Put the following argument into symbolic form. If the weather is favorable, we will work outside. The weather is favorable, therefore, we will work outside.		
35. What is the propositional function in 34 for which a truth-table can be constructed? (Conjoin the premises and indicate that the premises imply (⊃) the conclusion.)		
36. Construct the truth-table for the propositional function you wrote in 35.		

	Did not Check Back	Checked Back
III. Answer items 37-45 by doing any review of the instructional materials that is necessary.		
37. Put the following argument into symbolic form. If the weather is favorable, we will work outside. We do not work outside. Therefore, the weather is not favorable.	________	________
38. What is the propositional function in 37 for which a truth-table can be constructed?	________	________
39. Construct the truth-table for the propositional function you wrote in 38.	________	________
40. Put the following argument into symbolic form. If the weather is favorable, we will work outside. The weather is not favorable. Therefore we will not work outside.	________	________

	Did not Check Back	Checked Back
41. What is the propositional function in 40 for which a truth-table can be constructed?	________	________
42. Construct the truth-table for the propositional function you wrote in 41.	________	________
43. Put the following argument into symbolic form. If Jones wins the election, then we will either have a depression or war. But it is clear that there can be no war. Therefore, it is impossible for Jones to win the election.	________	________
44. What is the propositional function in 43 for which a truth-table can be constructed?	________	________
45. Construct the truth-table for the propositional function you wrote in 44.	________	________

SYMBOLIC LOGIC

PROPOSITIONAL CALCULUS - TRUTH TABLE CONSTRUCTION AND ANALYSIS

Mode of Presentation IV

INTRODUCTION: Three Kinds of Arguments

The purpose of logic is to establish criteria for valid arguments. Truth-tables provide such criteria. Truth-tables are used to classify arguments into one of the following three classes:

(1) Logically true - when the premises are true, the conclusion must be true.

(2) Contingently true - when the premises are true, the conclusion may or may not be true, depending upon particular circumstances.

(3) Logically false - when the premises are true, the conclusion must be false.

For example, the conclusion:

John is taller than Bill

must be true if the premises:

John is taller than Sam

and

Sam is taller than Bill

are both true. However, the conclusion:

Bill is taller than Sam

is not necessarily true when the two premises,

John is taller than Sam

and

John is taller than Bill

are both true. Therefore, this conclusion would be only contingently true.

As an example of a conclusion which contradicts the premises, the statement

Bill is taller than John

is necessarily false, when the premises:

John is taller than Sam

and

Sam is taller than Bill

are both true. That is, it follows from the premises, that John must be taller than Bill. The statement that Bill is taller than John is, therefore, necessarily false.

When the conclusion must be true, if the premises are true, the argument is said to be valid. All other arguments are invalid. That is, arguments in which the conclusion may be true when the premises are true and arguments in which the conclusion cannot be true when the premises are true are both considered invalid arguments.

Truth and Validity

It is important not to confuse truth with validity. An argument is either valid or invalid, but the premises and conclusion(s) are themselves either true or false. It is meaningless to say that a premise in an argument is valid, for this implies that it is a conclusion rather than a premise. In order to refer to statements in an argument without making a distinction between premises and conclusions, the term "proposition" is used to designate either a premise or a conclusion. A "proposition" is thus defined as any sentence which must be either true or false, but which cannot be both true and false: (a) below, is a proposition, but (b) and (c) are not.

(a) The cat is on the mat.

(b) The pain in my arm is blue.

(c) John is both present and absent.

Truth, then, is a property of propositions and validity is a property of arguments.

Propositional Calculus

The calculus of proposition, or propositional calculus, is a basic part of logic and the starting point for the study of symbolic logic. As the term implies, "propositional calculus" is the process of calculating using propositions as the basic unit. All that is required is that the truth-value (truth or falsity) be known. The subject-matter, or content, of the proposition is irrelevant.

Before truth-tables can be constructed for classifying arguments, the basic structural form of arguments must be examined. In order to do this, a method of symbolization must be adopted. This is accomplished by noting that propositions are linked by such words as "not", "and", "or", and "if -- then --". Thus, the problem of symbolization is divided into two parts (1) to define symbols which stand for propositions and (2) to provide suitable symbols for words which link or connect propositions into arguments. Therefore the following method is adopted:

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(1) Letters of the alphabet p, q, r, etc. which stand for propositions. These letters, as used in the propositional calculus, are known as propositional variables because they stand indifferently for any proposition.

(2) Words like "not", "and", "or", and "if--then--" which link the propositional variables into truth-functions; i.e., compound expressions like "not p or q" or "if p, then q" whose truth-values depend on the truth-values of their component propositions, are known as logical constants.

The logical constants are represented by symbols, thus:

For "not p", write "$\sim p$".

For "p and q", write "$p \cdot q$".

For "p or q", write "$p \vee q$".

For "if p, then q", write "$p \supset q$".

Since a proposition must be either true or false, and since when "p" is true, "not p" is false, and conversely, the following matrix represents all possibilities.

p	not p
true	false
false	true

When two variables are considered, the matrix must contain two columns and four rows.

p	q
true	true
true	false
false	true
false	false

That is, any combination of p and q must provide for all possible arrangements of "true" and "false".

If negations are included, the matrix contains four columns as well as four rows.

p	$\sim p$	q	$\sim q$
true	false	true	false
true	false	false	true
false	true	true	false
false	true	false	true

Taking two propositional variables and their respective possible truth-values:

p	q
true	true
true	false
false	true
false	false

A third column of truth-values can be defined for a combination of the two given variables. The first such combination defined will be for the logical constant "and" symbolized as "$\cdot$", known as the conjunctive function.

The Conjunctive Function

The conjunctive function is merely any two propositional variables linked by the conjunction "and". The problem is to ascertain the truth-values of two propositional variables taken conjunctively for all possible true-false combinations of the individual variables. For example, consider the two propositions:

$p =$ the cat is on the mat.

$q =$ the cat is drinking milk.

The compound proposition, "the cat is on the mat and the cat is drinking milk", (p • q) is true only when both p and q taken separately are true. In other words, the truth of each part is a necessary condition for the truth of the whole.

The truth-table for the conjunctive function is therefore the following:

p	q	(p • q)
true	true	true
true	false	false
false	true	false
false	false	false

Another example is the statement, "If it is Friday and the thirteenth, then you will have bad luck." For the conjunctive function; "Friday and the thirteenth" to be true, each part must be true, i.e., it must be true that it is Friday and also true that it is the thirteenth.

The Disjunctive Function

The next combination of p and q for which a truth-table must be defined is the disjunctive function p or q, symbolized by (p v q). In this case, the truth of either p or q is sufficient for the truth of the compound proposition (p v q). For example, let:

$p \equiv$ snow is falling

and

$q \equiv$ it is raining

Then, the truth-value of the compound proposition "it is raining or snowing" must be assessed. This is easiest done when the truth of (p v q) is made a condition for a subsequent statement. Consider, "If it rains or snows, then the streets will be slick". The truth of either p or q taken separately is sufficient for the truth of (p v q), that is, for saying that the streets will be slick. The only time (p v q) is false is when both p and q taken separately are both false. We can not say that the streets will be slick when it is neither raining nor snowing.

Thus, the truth table for the disjunctive function is of the following form:

p	q	(p v q)
true	true	true
true	false	true
false	true	true
false	false	false

Sometimes in ordinary discourse, the distinction between the conjunctive and disjunctive functions is not clear. Depending on circumstances, (p v q) may mean p or q but not both, and sometimes it may mean p or q and perhaps both. For example, if membership in a major political party is a requirement for holding a public office, being either a Democrat or a Republican would be a sufficient condition, but presumably a candidate could not be both. On the other hand, a person may qualify for voting rights if he either resides or owns property in a given district. In this case he may meet either or both of the requirements.

The case where p and q cannot both be true at the same time is known as the exclusive use of "or". When p and q can both be true, "or" is used inclusively. This dual usage of $(p \vee q)$ sometimes results in lack of clarity in the use of language, but it does not cause any trouble in truth table construction. In specific cases when $(p \vee q)$ cannot be true for both variables at the same time, the first row in the truth-table is merely interpreted to mean that both p and q *can* be true, but not necessarily at the same time.

Since the disjunctive function is true in all cases except when the two propositional variables are both false, a truth-table corresponding to the propositional function $(\sim p \vee q)$ can be constructed. Remembering that p must be false when p is true, and conversely, the following table is generated.

p	$\sim p$	q	$(\sim p \vee q)$
true	false	true	true
true	false	false	false
false	true	true	true
false	true	false	true

The function $(\sim p \vee q)$ is false only in the second row when $\sim p$ and q taken separately are both false.

A verbal instance of $(\sim p \vee q)$ is, "*Either* he is not a candidate *or* he will attend the convention." This statement is equivalent to, "If he is a candidate, then he will attend the convention". Another example is, "Either he did not rob the bank or he should be convicted". This statement is semantically equivalent to, "If he robbed the bank, then he should be convicted". Another instance is, "Either he was not hired or he is a college graduate". This statement means the same as "If he was hired, then he is a college graduate". Note that in each case a statement of the form ($\sim p \vee q$) is equivalent to another statement containing the same propositional variables, (p and q) but in the form "if p, then q" or ($p \supset q$). Therefore the truth-table for ($p \supset q$) must be the same as that for ($\sim p \vee q$).

The Implicative Function

The truth table for the implicative function is defined as follows:

p	q	($p \supset q$)
true	true	true
true	false	false
false	true	true
false	false	true

Note that in the conjunctive and disjunctive functions, the truth tables were justified by reference to actual usage in ordinary discourse. The truth-table for the implicative function was justified by noting an equivalence between the "if -- then --" form and the form "not p or q" which was already introduced. Instances of the four possible truth functions must then be given for the implicative function, but it is necessary to remember that the support for the particular truth-array given (true, false, true, false) is in the equivalence of ($p \supset q$) and ($\sim p \vee q$) rather than in a clear correspondance between the truth-table and instances in ordinary usage (as was the case with the first two functions.)

The best way to interpret the truth-table for the implicative function in ordinary usage is to note that the function (p ⊃ q) is false only when a true premise (p) leads to a false conclusion (q). The first and last rows merely show that the truth value is preserved. Thus, the implication is true both when a true premise leads to a true conclusion and when a false premise leads to a false conclusion. This leaves only the third row to be interpreted. Why should the compound function (p ⊃ q) be true when a false premise leads to a true conclusion? This problem can best be understood by examining the last two rows together. Regardless of whether the conclusion is true or false, if the premise is false the implicative function is true. We usually reason from premises known to be true. Hence, to begin with false premises appears strange at first. But this interpretation is often found in idiomatic English. An example of an idiomatic expression is "If he passes that examination, then I will eat my hat". Both p and q are intended to be false; i.e., the speaker does not expect the person to pass the examination and he does not expect to eat his hat. But he intends for the compound function to be interpreted as true. This is therefore, an instance of the last row in the truth-table. The assertion could be modified to "If he passes that examination, then I will eat my lunch", i.e., q is true in this instance. The intention is still that the compound function will be interpreted as true.

Instances of the last two rows are seldom encountered because the truth of the antecedent (premise) is usually a condition which must be met before any subsequent action can be taken. Then if the consequent (conclusion) is true, the entire function is considered true, and if the consequent is false, then $(p \supset q)$ is considered false. Nevertheless, for the purposes of logic, all four possibilities must be included.

The Equivalence Function

Just as propositions (p, q, s...) can be conjoined in the conjunctive function, compound propositions $(p \supset q)$, $(p \vee q)$, etc. can also be conjoined. The truth-tables for $(p \supset q)$ and $(q \supset p)$ can be constructed yielding the following table: (Numbers indicate order of computation).

(1)	(2)	(3)	(4)
p	q	$(p \supset q)$	$(q \supset p)$
true	true	true	true
true	false	false	true
false	true	true	false
false	false	true	true

Note that the second and third rows for $(q \supset p)$ are the reverse of the corresponding rows for $(p \supset q)$ because the antecedent and consequent have been interchanged.

When columns (3) and (4) are conjoined, remembering that the conjunctive function is true only when both parts are true, the 5th column is generated.

(1) p	(2) q	(3) $(p \supset q)$	(4) $(q \supset p)$	(5) $(p \supset q).(q \supset p)$
true	true	true	true	true
true	false	false	true	false
false	true	true	false	false
false	false	true	true	true

The function $(p \supset q).(q \supset p)$ translated into verbal form becomes "If p then q" and "if q then p". This is equivalent to saying "p if and only if q". For example, the statement, "I will go to New York if and only if it is Friday", means "If it is Friday, then I will go to New York, and if I go to New York, then it is Friday". Each is therefore both a necessary and sufficient condition for the other. The propositional variables are, hence, said to be equivalent and written in the form $(p \equiv q)$. This function is called the equivalence function and the truth-table is that computed in column (5) above.

p	q	$p \equiv q$
true	true	true
true	false	false
false	true	false
false	false	true

Other examples of the equivalence function are (1) If it is leap year, then February has 29 days, and (2) If John is a bachelor, then he is not married. Both of these statements can be reversed without altering their truth-value.

Logical Equivalence

It was shown that the expressions $(\sim p \vee q)$ and $(p \supset q)$ are equivalent in meaning. The truth-table derived for $(\sim p \vee q)$ was assigned to $(p \supset q)$. An attempt was made to show that the truth-table for $(p \supset q)$ reflects the way language is used in everyday speech. In general, it is difficult to comprehend what meaning is conveyed by a complicated propositional function. Thus the method of seeking propositional forms which are equivalent in meaning by examining instances of each is very limited in usefulness. Therefore, for logical purposes, two propositional functions are said to be logically equivalent if the corresponding truth-tables are the same. Consider the propositional function $\sim(p . \sim q)$, read "It is not the case that both p and not q". The truth table for this function is the following:

(1)	(2)	(3)	(4)	(5)
p	q	$\sim p$	$(p . \sim q)$	$\sim(p . \sim q)$
true	true	false	false	true
true	false	true	true	false
false	true	false	false	true
false	false	true	false	true

The truth table for $\sim(p . \sim q)$ is the same as that for $(p \supset q)$. Therefore $\sim(p . \sim q)$ is logically equivalent to $(p \supset q)$.

Scope of Logical Constants

In functions such as $\sim(p \cdot \sim q)$ it is clear that in the first instance the constant "$\sim$" controls or operates over the entire expression, while in the second instance "$\sim$" applies only to the propositional variable q. Also the "$\sim$" operates over only that part of the expression which is on the right of it. For example, in $(p \cdot \sim q)$ the "$\sim$" operates over or controls only the variable q. On the other hand, the constants "$\cdot$", "v", "$\supset$", and "$\equiv$" control those parts of the expression which appear on both sides of them. These constants are therefore called binary constants.

Propositional functions containing only binary constants and negations would be simple in form, $p \cdot q \vee \sim r \supset p$, but very difficult to interpret. The parts must be separated by the use of parentheses and brackets, "(", "[". The above ambiguous expression could become $[(p \cdot q) \vee (\sim r] \supset p$. Another alternative is $(p \cdot q) \vee (\sim r \supset p)$. Still another possibility is $p \cdot [(q \vee \sim r) \supset p]$. All of these propositional functions are different, but when parentheses and brackets are properly used, there is no ambiguity.

It is therefore necessary to understand how the logical constants are used in conjunction with parentheses and brackets. Consider the following compound propositional function:

(a) If it is Thursday or Friday, then I will go to town.

Letting T, F and G represent "Thursday", "Friday", and "go to town" respectively, the function is reduced to proper symbolic form:

$$(T \vee F) \supset G$$

Any other use of parentheses would be incorrect. For example the expression : $T \vee (F \supset G)$

translated into a verbal instance, becomes:

(b) Either it is Thursday or if it is Friday, then I will go to town.

Clearly (b) does not convey the same meaning or information as (a).

The proper use of logical constants with parentheses and brackets can be stated in two simple rules.

Rule I: The scope of "$\sim$" is confined to the propositional variable immediately succeeding it, except where "$\sim$" is followed by a left-handed parenthesis "(". In this case, the scope of "$\sim$" extends to the following parenthesis.

Example 1: In $\sim p \cdot q$, the scope of "$\sim$" is p.

Example 2: In $\sim [p \cdot (q \supset r) \vee s]$, the scope of "$\sim$" is the entire expression.

Example 3: In $\sim (p \cdot q) \vee (r \supset s)$, the scope of "$\sim$" is $(p \cdot q)$.

Rule II: If a binary constant is flanked by a propositional variable, the scope of the constant on that side is confined to the propositional variable. If a binary constant is flanked by a parenthesis or bracket, the scope of the constant extends to the corresponding parenthesis or bracket.

Example 1: In the expression $p \supset (q \vee r)$, the scope of "$\supset$" is the whole expression: i.e., p and $(q \vee r)$.

Example 2: In the expression:

$$[(p \supset q) \supset r] \supset [(r \supset p) \supset (s \supset p)]$$

the scope of the first "$\supset$" is p and q; the scope of the second "$\supset$" is $(p \supset q)$ and r; the scope of the third "$\supset$" is the whole expression, etc.

Three Types of Argument

Almost anyone can ascertain whether or not an argument is grammatically well formed but in all but the simplest instances, difficulty is experienced in determining if an argument is valid. For example, the conclusion:

all fish swim

certainly follows from the two premises

all fish live in water

and

all creatures that live in water swim

However, the conclusion:

Jones did not win the election

does not obviously follow from the premises:

(1) If Jones won the election, then prices will rise.
(2) If prices rise, then consumer spending decreases.
(3) Consumer spending does not decrease.

Truth-tables provide a method for determining:

(a) if the conclusion must be true when the premises are true (valid argument)
(b) if the conclusion may be true or false when the premises are true (invalid argument)
(c) if the conclusion must be false when the premises are true (invalid argument)

In the case of (a) the last columns constructed in the truth-table contains only T's. In (b) the last column contains both T's and F's and in cases described by (c) the last column constructed contains only F's.

Any propositional function for which a truth-table containing a final column of T's is obtained is therefore said to be logically true or a tautology. When the final column contains both T's and F's, the function is contingently true. And if a final column containing only F's is obtained, the function is logically false or contradictory.

For example the function:

$$[(p \supset q) \cdot p] \supset q$$

should be logically true because the two premises $(p \supset q)$ and p state that whenever p occurs, q can be expected to follow, and that as a matter of fact, p does occur. Therefore the conclusion, q, should logically follow. The truth-table is the following:

(1)	(2)	(3)	(4)	(5)
p	q	$(p \supset q)$	$(p \supset q) . p$	$[(p \supset q) . p] \supset q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The numbers indicate the order of computation: (3) was computed from (1) and (2), (4) was computed from (3) and (1), (5) was computed from (4) and (2). Note that in computing (5), column (4) represents the antecedent and column (2) the consequent. Since there are no rows in column (4) and (2) which contain a T for the antecedent and an F for the consequent, there are no F's in column (5). Therefore the function is logically true. Hence if someone states:

(1) If it snows, the streets will be slick
(2) It is snowing
(3) Therefore, the streets will be slick

he has given a valid argument.

On the other hand, if someone presents the argument:

(1) If it snows, the streets will be slick
(2) The streets are slick
(3) Therefore it is snowing

his reasoning is invalid because the streets may be slick for other reasons. However, the conclusion does not contradict the premises. Hence, the truth-table should show that the conclusion may or may not be true depending on the particular circumstances involved. That is, the last column should contain both T's and F's.

(1) p	(2) q	(3) $(p \supset q)$	(4) $(p \supset q) \,.\, q$	(5) $[(p \supset q) \,.\, q] \supset p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Column (5) contains one F. Therefore the argument is invalid.

Contradictory reasoning is illustrated by the following argument:

(1) If it snows, the streets will be slick
(2) It is snowing
(3) And the streets are not slick

This argument may be symbolized as:

$$(p \supset q) \,.\, (p \,.\, \sim q)$$

The truth-table for the function should yield a last column containing only F's.

(1) p	(2) q	(3) $\sim q$	(4) $(p \supset q)$	(5) $(p \,.\, \sim q)$	(6) $(p \supset q) \,.\, (p \,.\, \sim q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

The classification of propositions yields two important rules:

(1) Every logically false or contradictory expression is the negation of a logically true or tautologous expression and vice-versa.

(2) Every logically true expression is logically equivalent to every other logically true expression; and similarly every contingent expression is logically equivalent to every other contingent statement.

Construction and Application of Truth Tables

When constructing truth-tables, begin at the left and move toward the right. Start with the individual variables, and proceed next to the constants of narrowest scope. Numbered headings may be used to indicate order of computation. Whenever it is necessary, for convenience, to rewrite a column, the same numbered heading should be used.

For example, the truth table for the propositional function:

$$[(p \supset q) \;.\; {\sim}q] \supset {\sim}p$$

could be constructed as follows:

(1)	(2)	(3)	(4)	(5)	(6)
p	q	$\sim p$	$\sim q$	$(p \supset q)$	$(p \supset q) \cdot \sim q$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	F	T	F
F	F	T	T	T	T

(6)	(7)	(3)
$[(p \supset q) \cdot \sim q]$	$\supset$	$\sim p$
F	T	F
F	T	F
F	T	T
T	T	T

Column (7) contains only T's. Hence, the propositional function in question is logically true or a tautology. Note that columns (6) and (3) were recopied for ease in computing column (7).

Only arguments as propositional functions involving two variables have been discussed. Most arguments, of course, contain more than two variables. Consider the following:

(1) If it rains, the streets will be slick.
(2) If the streets are slick, there will be accidents.
(3) Therefore, if it rains, there will be accidents.

Letting the letters R, S, and A represent "rains", "slick", and "accidents", respectively the argument is reduced to the form:

$$[(R \supset S) \cdot (S \supset A)] \supset (R \supset A)$$

The truth table is constructed as follows:

(1)	(2)	(3)	(4)	(6)	(5)	(8)	(7)
R	S	A	$[(R \supset S)$	.	$(S \supset A)]$	$\supset$	$(R \supset A$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Column (8) contains all T's, therefore, the argument is valid.

Note that when an argument involves three propositional variables the truth table must contain eight rows in order to insure that all possible permutations of T's and F's will be included for any two given variables.

Illustrative Examples

1. The following arguments may be checked for validity using truth tables:

If A is elected then B will resign.
If C is elected, then B will not resign.
Therefore, if A is elected, then C will not be elected.

The premises are symbolized as follows:

$$(A \supset B), (C \supset \sim B)$$

The conclusion is the following:

$$(A \supset \sim C)$$

The entire argument is therefore represented as the following propositional function:

$[(A \supset B) \, . \, (C \supset \sim B)] \supset (A \supset \sim C)$

The truth-table is constructed as follows:

(1) A	(2) B	(3) C	(4) ~B	(5) ~C	(6) (A⊃B)	(7) (C ⊃ B)
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	T	F	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

(6) [(A ⊃B)	(9) .	(7) (C⊃~B)]	(10) ⊃	(8) (A⊃~C)
T	F	F	T	F
T	T	T	T	T
F	F	T	T	F
F	F	T	T	T
T	F	F	T	T
T	T	T	T	T
T	T	T	T	T
T	T	T	T	T

Since the last column computed (10) contains only T's the argument is valid.

2. If there is an election, the government will not remain in power. Either the government will remain in power or there will be a coup d' etat. There will not be a coup d' etat. Therefore, there will not be an election.

Letting the letters E, P, and C represent "election", "power", and "coup", respectively, the premises of the argument become:

$$[E \supset \sim P), \ (P \vee C), \text{ and } \sim C$$

The entire argument in symbolic form is thus:

$$([(E \supset \sim P) \,.\, (P \vee C)] \,.\, \sim C) \supset \sim E$$

The validity of the argument is then tested by truth-table analysis:

(1) E	(2) ~E	(3) P	(4) ~P	(5) C	(6) ~C	(7) (E⊃~P)	(8) (P v C)
T	F	T	F	T	F	F	T
T	F	T	F	F	T	F	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F

(7) ([(E⊃~P)	(9) .	(8) (P v C)]	(10) .	(6) ~C))	(11) ⊃	(2) ~E
F	F	T	F	F	T	F
F	F	T	F	T	T	F
T	T	T	F	F	T	F
T	F	F	F	T	T	F
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	F	F	F	T	T	T

The final column contains all T's. Therefore the argument is valid.

3. If there is a change in government, then confidence in the dollar will not be restored. If confidence in the dollar is not restored, then imports will be restricted. Therefore, if imports are not restricted, there will not be a change in government.

Let the symbols "Ch", "Co", and "R" represent "change", "confidence", and "restored" respectively. Then the premise becomes:

$$(Ch \supset \sim Co) \text{ and } (\sim Co \supset R)$$

The conclusion is:

$$(\sim R \supset \sim Ch)$$

The truth-table is constructed as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(9)	(8)
Ch	~Ch	Co	~Co	R	~R	(Ch ⊃ ~Co)	.	(~Co ⊃ R)
T	F	T	F	T	F	F	T	F
T	F	T	F	F	T	F	F	T
T	F	F	T	T	F	T	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	T	F	T	T	T
F	T	T	F	F	T	T	T	T
F	T	F	T	T	F	T	T	T
F	T	F	T	F	T	T	F	F

(7)	(9)	(8)	(11)	(10)
[(Ch ⊃ ~Co)	.	(~Co ⊃ R)]	⊃	(~R ⊃ ~Ch)
F	F	F	T	T
F	F	T	T	F
T	T	T	T	T
T	F	F	T	F
T	T	T	T	T
T	T	T	T	T
T	T	T	T	T
T	F	F	T	T

Column (11) computed from column (9) and column (10), contains only T's. Therefore the argument is valid.

4. If Jones resigns, the party will split and there will be an election. If the party splits and there is an election, the international situation will not improve. But the international situation does improve. Therefore, Jones won't resign.

Letting the letters R, S, E, and I represent "resign, "split", "election" and "international situation" respectively, the premises become:

$$R \supset (S \,.\, E) \quad , \quad (S \,.\, E) \supset {\sim}I \quad \text{and } I$$

and the conclusion becomes:

$$\sim R$$

The entire argument, in symbolic form is thus:

$$\big([R \supset (S \,.\, E)] \,.\, [(S \,.\, E) \supset {\sim}I] \,.\, I\big) \supset {\sim}R$$

The truth-table is constructed as follows: (note: that since there are four separate propositional variables, there must be sixteen rows in the truth table.)

(1) R	(2) $\sim R$	(3) S	(4) E	(5) I	(6) $\sim I$	(7) $(S \,.\, E)$	(8) $[R \supset (S \,.\, E)]$
T	F	T	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	T	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F
T	F	F	T	F	T	F	F
T"	F	F	F	T	F	F	F
T	F	F	F	F	T	F	F
F	F	T	T	T	F	T	T
F	T	T	T	F	T	T	T
F	T	T	F	T	F	T	T
F	T	T	F	F	T	F	T
F	T	F	T	T	F	F	T
F	T	F	T	F	T	F	T
F	T	F	F	T	F	F	T
F	T	F	F	F	T	F	T

(8)	(10)	(9)	(11)	(5)	(12)	(2)
$([R \supset (S \,.\, E)]$	$.$	$[(S \,.\, E) \supset \sim I]$	$.$	$I)$	$\supset$	$\sim R$
T	F	F	F	T	T	F
T	T	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
F	F	T	F	T	T	F
F	F	T	F	F	T	F
T	F	F	F	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	T	T	T	T	T	T
T	T	T	F	F	T	T

Column (12) contains only T's. Therefore the argument is valid.

5. If Jones is guilty or the police are suspicious, then either Jones will bribe the police or he will remain in town. Therefore, if Jones remains in town or the police are suspicious, it is false that he is both guilty and is not bribing the police.

Using the symbols G, S, B, and R, to represent "guilty", "suspicious", "bribe", and "remain", respectively, the premise becomes:

$$(G \vee S) \supset (B \vee R)$$

The conclusion is as follows:

$$(R \vee S) \supset \sim (G \,.\sim B)$$

The entire argument is thus:

$$[(G \vee S) \supset (B \vee R)] \supset [(R \vee S) \supset \sim (G \,.\sim B)]$$

Since there are four propositional variables, the truth-table contains sixteen rows.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
G	S	B	~B	R	~R	(G v S)	(B v R)
T	T	T	F	T	F	T	T
T	T	T	F	F	T	T	T
T	T	F	T	T	F	T	T
T	T	F	T	F	T	T	F
T	F	T	F	T	F	T	T
T	F	T	F	F	T	T	T
T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
F	T	F	T	F	T	T	F
F	F	T	F	T	F	F	T
F	F	T	F	F	T	F	T
F	F	F	T	T	F	F	T
F	F	F	T	F	T	F	F

(7)	(9)	(8)	(14)	(10)	(13)	(12)	(11)
[(G v S)	⊃	(B v R)]	⊃	[(R v S)	⊃	~	(G . ~B)]
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	T	F	F	T	F	F	T
T	F	T	T	T	F	F	T
T	T	T	T	T	T	T	F
T	T	T	T	F	T	T	F
T	T	F	F	T	F	F	T
T	F	T	T	F	T	F	T
T	T	T	T	T	T	T	F
T	T	T	T	T	T	T	F
T	T	F	T	T	T	T	F
T	F	T	T	T	T	T	F
F	T	T	T	T	T	T	F
F	T	T	T	F	T	T	F
F	T	F	T	T	T	T	F
F	T	T	T	F	T	T	F

The truth-table for the last column contains two F's, Therefore the argument is invalid.

EXPERIMENTAL PROJECT TEST

INSTRUCTIONS

On your answer sheet: PRINT:

NAME
GRADE (freshman, sophomore, junior or senior)
SEX
DATE OF BIRTH
TODAY'S DATE
AGE
SCHOOL (put INSITE or F100)
IDENTIFICATION NUMBER
CAMPUS ADDRESS

Read each of the following items carefully. Using a regular pencil, mark in the appropriate answer space, the choice which best corresponds to the stated conditions. Try to answer all the items, but you will be penalized for guessing.

Make sure that your marks are heavy and black. Erase completely any answers you wish to change.

There is no time limit but note on your answer sheet (beside CITY) the number of minutes you needed to finish the test.

1. **Truth tables provide a method for:**

   (a) Separating valid and invalid arguments.

   (b) Classifying arguments according to whether they are logically true, contingently true, or contradictory.

   (c) Both (a) and (b).

   (d) Neither (a) nor (b).

2. **Which of the following is not a proposition?**

   (a) Today is Thursday.

   (b) Monday is Thursday.

   (c) Monday is the first day of the week.

   (d) Today is the first day of the week.

3. **By inspection, the argument,**

   All rocks are hard
   Limestone is hard
   Therefore, limestone is rock

   is:

   (a) A tautology (conclusion must be true if premises are true).

   (b) A contradiction (conclusion must be false if premises are true).

   (c) Both (a) and (b)

   (d) Neither (a) nor (b)

4. **Which of the following best defines a proposition?**

   (a) Any statement containing both a subject and a predicate.

   (b) Any statement which is either true or false, but not both.

   (c) Any statement which refers to physical objects.

   (d) Any statement which is not meaningless.

5. **A logical constant is:**

   (a) A proposition of which the truth value never changes.

   (b) A word or words connecting two or more propositions.

   (c) The property of always being true or always being false.

   (d) Symbolized by the letters (p, q, r, etc.)

6. In the compound propositional function symbolized by $(p \,.\, q)$ the truth value of the function is:

(a) undefined

(b) determined by the truth-values of the individual variable, p and q.

(c) Always false

(d) Always true

7. Four separate truth functions are defined in the propositional calculus. These are: conjunction, disjunction, implication and:

(a) negation

(b) contraversion

(c) equivalence

(d) inversion

8. If a proposition, p, is true, then the denial $(\sim p)$ is:

(a) always true

(b) always false

(c) could be true

(d) could be false

9. Given the proposition, "The cat is on the mat", the denial is:

(a) Two cats are on the mat.

(b) The mat is on the cat.

(c) Sometimes the cat is on the mat.

(d) The cat is not on the mat.

10. The proposition, "If today is Monday, then I will go to work" is of the form.

(a) $(p \supset q)$

(b) $(p \,.\, q)$

(c) $(p \vee q)$

(d) $(p \equiv q)$

11. Which of the following two propositional forms are equivalent ?

(a) $(p \supset q)$ and $(\sim p \supset \sim q)$

(b) $(p \supset q)$ and $(\sim q \supset \sim p)$

(c) $(p \supset q)$ and $(\sim p \,.\, q)$

(d) $(p \supset q)$ and $(p \vee \sim q)$

12. Given the two premises:

Either the sun is shining or it will rain
But the sun is not shining

which of the following conclusions is determined.

(a) Therefore it will rain

(b) Therefore it will not rain

(c) Therefore it will probably rain

(d) Therefore it will probably not rain.

13. Which of the following is not a properly symbolized propositional function ?

(a) $(p \supset q) \vee r$

(b) $(p \,.\, q) \supset r$

(c) $(p \vee r) \,.\, q$

(d) $p \vee q \supset r$

14. The proposition: If anyone is both a high school graduate and 21 years of age, then he qualifies for the job.

is symbolized as:

(a) $p \supset (q \,.\, r)$

(b) $(p \,.\, q) \supset r$

(c) $p \supset (q \supset r)$

(d) $p \vee q \supset r$

15. If someone says: the table is either brown or blue or red and
then adds, but it is clear that it is not red.

you would then expect him to say:

(a) then it must be either brown or blue

(b) then it is not brown or blue

(c) then it must be brown or not blue

(d) then it must be brown and blue

16. If the expressions (p v q) and (q v p) are logically equivalent, then:

(a) any verbal instances of each must be identical

(b) the truth-tables for each will be the same

(c) the truth-table for each must contain all T's

(d) the truth-table for each must contain all F's.

17. If the propositional function: $[(p \supset q) \,.\, q] \supset p$ is truth-tabled and the last column computed contains both T's and F's then:

(a) the truth-table is incomplete

(b) the truth-table is incorrectly calculated

(c) the symbolized argument is invalid

(d) the symbolized argument is meaningless

18. In the argument: Jones is either right or he will be elected and serve
But he does not serve
Therefore Jones is not right

the conclusion:

(a) follows logically from the premise

(b) follows contingently from the premises

(c) is true whether the premises are true or not.

(d) is false whether the premises are true or not.

19. In the argument: All students study
All teachers study
Therefore, all teachers are students

the conclusion may be true for part of the class of teachers, but certainly is not true for all. Therefore, in this kind of argument, the conclusion is said to be:

(a) contingently true

(b) conceptually true

(c) selectively true

(d) partially true

20. Every contingently true statement is said to be logically equivalent to every other contingently true statement. This means that:

(a) Any two statements which are contingently true mean the same thing

(b) The class of contingently true statements cannot be distinguished from the class of logically true statements.

(c) Classified according to kinds of truth, there are three kinds of propositions.

(d) Logicians have not completed their work.

21. In the argument: All fish live in water
All whales are fish
Therefore, all whales live in water.

(a) The argument cannot be valid because one premise is false.

(b) The conclusion is true, so it does not matter whether the premises are true or not.

(c) Validity cannot be determined until we check on the truth of each premise.

(d) The argument is valid which shows that a valid argument can contain false premises.

22. The truth-table for the conjunctive function (p . q) was:

(a) false for two cases and true for two

(b) false for three cases and true for one.

(c) false for one case and true for three

(d) false for all cases.

23. If both members of the conjunction (p . q) are false, the entire function is false. This is because:

(a) the meaning of 'conjunction' is so defined.

(b) we use language this way in every day affairs

(c) when p is false, q is automatically also false

(d) both p and q cannot both be true at the same time.

24. Given the two statements: If it is both Friday and the thirteenth,
Then I will have bad luck
But it is neither Friday nor the thirteenth.

(a) It is impossible that I can have bad luck

(b) I am sure to have bad luck

(c) I can expect to have bad luck part of the time

(d) I have no good reason to expect either bad or good luck

25. The scope of the logical constant "$\supset$" is defined as:

(a) That part of the symbolized expression over which the constant controls or operates.

(b) That part of the symbolized expression which preceeds the constant

(c) That part of the symbolized expression which follows the constant

(d) That part of the symbolized expression included between parentheses.

26. In the propositional function: $(p \vee q) \supset \sim q$ the scope of "$\supset$" is:

(a) p

(b) $\sim q$

(c) $(p \vee q)$

(d) the entire expression

27. The statement: If an election is held, then either A or B or C will win.
is symbolized as:

(a) $E \supset (A \vee B) \vee C$

(b) $E \supset A \vee (B \vee C)$

(c) $E \supset A \vee B \vee C$

(d) Either (a) or (b) or (c)

28. Brackets and parentheses are used to:

(a) Introduce ambiguity into the expression symbolized.

(b) Separate the expression into parts over which the logical constants operate.

(c) Separate propositional variables from logical constants.

(d) Separate lower case letters from capital letters.

29. Which of the following expressions are equivalent?

(a) $(\sim p \vee q)$ and $\sim (p \vee q)$

(b) $\sim (p \vee q)$ and $(\sim p \vee \sim q)$

(c) $\sim (p \,.\, q)$ and $(\sim p \vee \sim q)$

(d) $\sim (\sim p \vee \sim q)$ and $(p \vee q)$

30. If someone says, "I am not eating and I am not sleeping", an equivalent statement would be:

(a) It is false that I am either eating or sleeping.

(b) It is true that I am eating, but false that I am sleeping.

(c) It is false that I am eating, but true that I am sleeping.

(d) It is false that I am eating and sleeping.

31. The statement: It is false that today is Monday
is symbolized as:

(a) $\sim M$

(b) $\sim M \supset F$

(c) $\sim M \vee F$

(d) $F \vee \sim M$

32. The statements: All unicorns have one horn.
It is false that unicorns have either no horns or more than one horn.
No unicorn has no horns or more than one horn.
If an animal is a unicorn, then it has one horn.

are:

(a) equivalent

(b) meaningless

(c) all true

(d) all false

33. If it is false that a surface is both red and green all over at the same time, then:

(a) It may be either red or green, but not both.

(b) It must be some other color than red and green

(c) It is necessarily a combination of the two colors.

(d) It is therefore transparent.

34. Which of the following propositional functions are defined as equivalent?

(a) $(p \supset q) \,.\, (q \supset p)$ and $p \equiv q$

(b) $(p \supset q)$ and $(q \supset p)$

(c) $(p \vee q)$ and $(p \,.\, {\sim} q)$

(d) $(p \vee q)$ and $(q \vee p)$

35. Which of the following expressions must be equivalent?

(a) $(p \,.\, q)$ and $(p \vee q)$

(b) $(p \supset q) \supset r$ and $p \supset (q \supset r)$

(c) $(p \supset p)$ and $(q \supset q)$

(d) $p \supset (q \vee r)$ and $(p \supset q) \vee r$

36. **Which two of the following propositional functions are logically equivalent?**

1. $[(p \supset q) . p] . \sim q$
2. $[(p \supset q) . \sim p] \supset \sim q$
3. $[(p \supset q) . q] \supset p$

(a) 1 & 2

(b) 1 & 3

(c) 2 & 3

(d) all are logically equivalent

37. **Given, the following argument:** **If the Republicans win, Jones will be elected. Jones is either defeated or he appoints his relatives to public offices. Therefore, if the Republicans win, Jones will appoint his relatives to public offices.**

Letting R, E, and A represent "Republicans win", "Jones is ELECTED **[illegible], and "relations appointed" respectively, which of the following symbolic expressions represents the two premises and conclusion?**

(a) $[(R \supset E) . (\sim E \vee A)] \supset (R \supset A)$

(b) $[(R \supset E) . (R \supset A)] \supset (\sim E \supset A)$

(c) $[(R \supset A) . (\sim E \supset A)] \supset (\sim E \supset R)$

(d) $[(R \supset \sim E) \vee (\sim E \vee A)] \supset (R \supset A)$

38. **In the argument for #37,**

(a) the conclusion follows logically from the premises.

(b) the conclusion contradicts one of the premises.

(c) the conclusion contradicts two of the premises.

(d) the conclusion does not follow logically from the premises.

39. **If a truth-table is constructed for the argument in #37, the last column would correspond to which of the following propositional functions?**

(a) $\supset$

(b) $\vee$

(c) $.$

(d) $\equiv$

40. How many propositional variables are in the argument given in #37?

(a) two

(b) three

(c) four

(d) five

41. A truth-table for the argument in #37 would contain _____ rows.

(a) four

(b) six

(c) eight

(d) ten

42. The truth-table for the implicative function is T F T T. The only time the function is false is when the antecedent is true and the consequent is false. Thus, given the truth-tables (T T F F) and (F T F T) for R and R and E respectively in #37, the truth-table for (R ⊃ E) would be:

(a) T F T T

(b) T T F T

(c) F T T T

(d) F F F T

43. In constructing truth-tables, care must be taken to insure that all possible T-F combinations are included for each function represented. This is done by:

(a) randomizing the order of each column.

(b) presenting each column in an ordered sequence of T's and F's.

(c) making sure that each column contains an equal number of T's and F's.

(d) making sure that T's and F's correspond to the truth or falsity of the premises involved.

44. The first four rows of a truth-table for an argument containing three propositional variables will be which of the following sequences?

(a) T T T T

(b) T T F F

(c) T F T F

(d) F F F F

45. In the argument:

War always leads to peace
And peace always leads to prosperity
Therefore, war leads to prosperity

the fourth column of the truth-table will correspond to which of the following variables or constants ?

(a) prosperity

(b) peace

(c) ⊃

(d) v

46. If someone argues that all politicians are honest because all politicians are citizens and there is no citizen that is not honest, the argument contains ________ variables.

(a) three

(b) four

(c) five

(d) six

47. Which of the following best applies to the argument given in #46 ?

(a) The last premise cannot be translated into one of the standard propositional forms.

(b) The truth of the premises must be established before a truth-table can be constructed.

48. In the argument in #46, the conclusion and first premise are represented symbolically as:

(a) $(P \supset H) \vee (P \supset C)$

(b) $(P \supset C) \supset (P \supset H)$

(c) $(P \supset P) \supset (C \supset H)$

(d) $(C \supset P) \supset (P \supset H)$

49. Consider the following two statements:

All responsible citizens work for the democratic ideal.
All who work for the democratic ideal are worthy citizens.

If these two statements are taken as premises in an argument, the logical constant connect them would be:

(a) ⊃

(b) ≡

(c) v

(d) .

50. Which of the following statements represents the conclusion(s) for the argument in #49?

(a) All responsible citizens are worthy citizens.

(b) If a person is not worthy, then he is not responsible.

(c) both (a) and (b)

(d) neither (a) nor (b)

51. A truth-table for the argument in #49 would contain _____ columns.

(a) five

(b) six

(c) seven

(d) eight

52. If the truth-table for a propositional variable (r) is
T F T F T F T F, the truth-table for (~ r) is:

(a) F F F F T T T T

(b) F T F T F T F T

(c) T T F F T T F F

(d) F F T T F F T T

53. An argument containing the variable (r) in #52 must contain _______ variables.

(a) two

(b) three

(c) four

(d) five

54. The truth-table for the argument referred to in #53 would contain __________ columns.

(a) eight

(b) six

(c) either six or eight

(d) The information is insuffient to tell.

55. If the last column constructed in a truth-table contains all T's we know that:

(a) the conclusion must be true.

(b) the conclusion is true most of the time.

(c) the conclusion must be true when the premises are true.

(d) if the premises are false, the conclusion must be false.

56. If the last column in a truth-table contains both T's and F's, we know that:

(a) truth is a matter of individual opinion.

(b) the conclusion and one premise must be reversed to have a valid argument.

(c) the conclusion may or may not be true.

(d) falsity is a matter of individual opinion.

57. Given the statement: If Jones is sick, then either he will call the doctor or stay in bed.

symbolized as: $S \supset (D \vee B)$

which of the following represents the order in which the columns of the truth-table would be constructed?

(a) S, D, B, ⊃, and v.

(b) S, D, B, v, and ⊃.

(c) D, B, S, ⊃, and v.

(d) D, B, S, v, and ⊃.

Truth-tables are given below for the four binary logical constants.

p	q	(p . q)	(p v q)	(p ⊃ q)	(p ≡ q)
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Use this table for items 58, 59, and 60.

58. In the argument: If Jones is sick, then he will either call
the doctor or stay in bed.
But he does not stay in bed
Therefore he is not sick.

What is the truth-table for (D v B)?

(a) T T T T T T T T

(b) T T T F T T T F

(c) T F T T T F T T

(d) F F T F F F T F

59. In the argument given in #58, what is the truth-table for:

$S \supset (D \vee B)$?

(a) T T T T F T T T

(b) F F F F F F F F

(c) T T T F T T T T

(d) T F T T F T F F

60. In the argument given for #58, what is the truth-table for:

$[S \supset (D \vee B)] \cdot \sim B$?

(a) F F T T F F T T

(b) T T T F T T T F

(c) T F T T F T T F

(d) F T F F F T F T

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Cine-Psychometry

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Cine-Psychometry[1]

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Many human decisions are made within contexts which are not static, not principally verbal, and not characterized by the presence of print. This would be a trivial observation if it were not for the dominant character of existing "intelligence" tests. In view of their character, however, we are entitled to wonder whether some significant human capacities for intelligent decision making have not been overlooked, not only in the great majority of marketed tests, but also in those intended principally for research use. Such possibilities appear to deserve greater attention than they have received during the sixty years of active work on the nature of intellect.

Although a recognition of variety in intellectual functioning can be easily demonstrated by reviewing prior work on human intelligence, the liberation of testing from print and, virtually, from printed language has not been immensely successful. Somewhat overstated, the point is that human intelligence has been

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regarded implicitly as a captive of print, of language, or of both. A broader view would be to acknowledge that past work has sampled incompletely from the range of human decisions by which the aspects of intellect are revealed. The purpose of this paper will be to review some earlier studies which, as the authors see them, reflect a broader view. In particular, it will review and seek implications from studies which have employed the medium of the motion picture (more simply, the "film") in psychological testing.

Cine-psychometry is not to be regarded as the ultimate answer to the expanded study of human intellectual variety, its nature, and its measurement, yet film is a medium with real and apparent capabilities, some of which relate clearly to the requirements of psychological testing. These have been occasionally recognized, thus saving tests in film form from complete obscurity, but the occasions have been rare and the resulting reports constitute little more than some spots scattered in the literature. If the reader wonders why extensive work in film testing has not developed, a partial answer is perhaps contained in the observation that educators and psychologists with interest in film have focused on demonstrating its applicability to instruction, thus overlooking or ignoring its applicability to those communications which we know as tests. This observation, however, is only a partial explanation since fair numbers of professionals have devoted themselves to the study of intelligence, not to film; they too have ignored the prospects of film testing with remarkable consistency.

GENERAL POTENTIALS

In considering film and its candidacy for use in psychological testing, it is helpful, first, to consider the points set forth in the first paragraph above. In addition, consideration must be given to the characteristics of film (or televised film or videotape) and their adaptability to testing purposes. Some of the relevant characteristics, although not an exhaustive nor mutually exclusive set, are the following: (1) Film (and television) provide for the manipulation and control of within-item exposure time and item response time intervals, as well as the customary manipulation-control of total time limits. (2) Film can sequence stimuli within an item, thus providing not only a fixed exposure sequence, but also establishing pace, rhythm, and/or motion. (3) Film can present complementary or competitive material visually and auditorially. (4) It can simulate many of the conditions of "real" decision situations. (5) Through the use of kinesic, ideographic, or cinematographic conventions, it can pose problems that minimize or eliminate the use of written or spoken language. (6) It can employ color as cues in the statement of problems. (7) It can, admittedly with some difficulty, approximate the visual third dimension and can utilize depth cues either to add complexity to conventional problems or more simply to test abilities related to depth judgments. (8) Film can not only encompass and present the range of item stimulus materials suggested above, but it can also accomplish several functions of the test administrator, while providing excellent control or standardization of testing conditions. (9) Finally, since film controls the pace of stimulus presentation, it offers the possibility of detailed and synchronized response recordings, including the indexing of response latencies, GSR's, EEG's, and other varieties of potentially helpful yet normally unmanageable response.

In part, the film's use in testing can originate in practical needs. The most apparent of these is the need to control and to standardize the conditions of testing. Here, although precise control of individual item exposure and response intervals may be a mixed blessing, the standardization of the form, content, and pacing of test instructions must be regarded as consistently desirable. If scores are to be useful, they must be derived under conditions that are comparable for all examinees; this comparability can be assured by means of the detailed and fully repeatable conditions which film-recorded directions provide. A second practical possibility stems from the increasing incidence of film and televised teaching, both of which may place students at some distance from their "teachers." In such settings, it may be desirable to have means for testing student progress which, like the teaching, can be managed from a distance. There is some possibility that film and/or television can contribute to the more effective management of testing in remote locations. It may also be true that the effects of audiovisual instruction can best be assessed with measures presented in the same form.

A third area of practical need relates to the testing of special, atypical populations of examinees. Two obvious examples are those populations that bear language handicaps stemming either from deficiencies in hearing or from deprived cultural backgrounds. The abilities of such populations cannot be adequately assessed with instruments that rely heavily on written or, perhaps, on spoken language. To meet such needs, special film tests could conceivably be prepared which would minimize the role of acquired language skills and would more adequately reflect the other abilities represented in such disadvantaged groups.

In addition to the practical needs and the related opportunities for developing serviceable film tests, similar tests are also promising as a further means toward the improved understanding of human intellectual organization. The likelihood of such contributions has been mentioned, directly or indirectly, by several of the most prominent figures in psychology. An indirect mention may be seen in Thurstone's discussion of spatial visualization ability (44, p. 404):

> The tests by which this factor has been identified all involve the visualizing of objects which are stationary. In dealing with mechanical problems, one must be able to think of objects in motion. Their relative motions have definite restrictions that are studied in a separate discipline known as kinematics. We might therefore add another psychological hypothesis, namely, that there exist one or more abilities that are revealed in the ability to think about solid objects in motion as distinguished from thinking about them when they stay still.

Guilford has also recognized some of the potentials of tests in film form. In his discussion of decisions which precede test construction, he wrote (19, p. 414): "... one's approach and operations ... will also depend upon the medium of the test, whether it is of printed form, of apparatus form, or of motion picture form." Later, Guilford suggested (20, p. 396):

> One error possibly made thus far in conventional research on empathy is too much dependence on verbal material. It may be true that there is much translating back and forth from one kind of material to another on the part of individuals in everyday life, and this would probably extend to testing operations. But there must be limitations to this. At any rate, it would be well to seek types of material that emphasize behavioral content, and this may often require motion-picture presentations.

In Guilford's discussion (21) of the abilities within the memory column of his Structure of Intellect model, he offers further indications that tests in film form could play a helpful role. He wrote:

> The area of memory abilities has been explored less than some of the other areas of operation, and only seven of the potential cells of the memory matrix have known factors in them. ... [p. 472] Considering the blank rows ... we should expect to find abilities also to remember

> classes, transformations, and implications, as well as units, relations, and systems. [p. 473]

The perceived relevance of the above is not only that we apparently know less concerning memory abilities than of other abilities, but also that the further study of memory requires instruments or tests that will not adapt readily to the medium of print. Quite likely, the pace and the sequence of item stimulus material will need to be controlled; the exposure interval should also be controllable within a range, probably, from the tachistoscopic to the extended exposure; and the interval between exposure and examinee recall or recording should also be controllable. Film offers the opportunity for these controls and others. Tests in printed form accomplish these things crudely, if at all.

The broadest arguments in behalf of film tests are those set forth by Gibson (17). He not only recognized many reasons for investigating such tests, but also contributed more than any other to their development. In discussing apparent parallels between human mental functions and motion pictures, he wrote (p. 20):

> The functions which will be exemplified are directly or indirectly related to the characteristics of motion pictures, as distinct from other modes of presentation . . . namely motion, sequence, pacing, and realism. These characteristics presumably have their psychological counterparts. Human behavior, and the capacities latent in it, also involves motion, order, tempo, and the experience of reality. It is reasonable to suppose, therefore, that the motion picture makes available to the test designer not only a special method of measuring known factors of human ability, but also gives him access to new and unnamed functions not accessible to conventional methods of test construction.

A further contribution that cine-psychometry might make relates to Cronbach's (11) argument concerning the need for unification of correlational and experimental psychology. Differential psychologists have tended to leave implicit the fact that performance on a single test item involves psychological processes occurring within an interval of time. Since only total testing time is controllable on group-administered printed tests, item performances are treated essentially as instantaneous responses, summable, as "bits of ability," to a total test score. While correlations among such scores, and factors derived from them, are valuable as descriptions of functional relationships existing within or among cognitive processes, they are *atemporal*. By contrast, experimental analysis of cognitive processes usually involves as a central concern hypotheses regarding the temporal-sequential nature of the process's functioning components. The degree of control which film permits over the presentation of within-item stimuli may permit a closer examination of the information-processing aspects of test performance and, hence, may provide a "bridge" between experimental and correlational interests without compromising the objectives of either.

In short, present interest in cine-psychometry developed from a consideration, first, that human abilities and human decision making are frequently demonstrated in settings that are not principally static, verbal, or dominated by print. Secondly, since the motion picture medium is complex, controllable, and versatile, it offers obvious possibilities for use in the communications which we know as tests. The use of film seems practical and, if Thurstone, Guilford, or Gibson are correct, perhaps uniquely appropriate in the study of ability dimensions which do not lend themselves to other forms of testing. Perhaps, also, the use of film may aid in coordinating experimental and correlational analysis of mental functioning.

A SHORT HISTORY OF PSYCHOMETRIC FILM

The general psychological literature for the present century is sprinkled with varied applications of film to satisfy research purposes. Cinematography and psychometry have been associated sporadically since the infancy of each, and the first known mention of film in testing is associated with the old "fidelity of report" experiment. According to Whipple (50), the idea that accuracy in reporting details of a witnessed event could serve as an index of intelligence was originally contributed by Binet, in his 1900 *La Suggestibilité*. The need for objectivity and reproducibility soon suggested motion picture recording and presentation of live-action stimulus material. For clinical use, Boas (3), in 1909, described filmic *vs.* verbal presentation of short stories to mental defectives and the use of *Ss'* recount of the story for diagnostic purposes. In 1912, Boring (4) brought fidelity of report of motion picture experience into the experimental laboratory, where it has since been occasionally used. Conrad (10) has summarized a series of investigations concerned with fidelity of report. Similar applications of motion pictures have also been made by Metfessel and Warren (30) and by Ray (33).

Somewhat tangential to the studies on fidelity of report, others on gestural communication have made use of live action film. Carmichael, Roberts, and Wessell (7) reported a study on judgment of the meaning of manual expressions; they used both still and motion picture presentation. The use of film in testing lip reading proficiency was described by Utley (47). Another noteworthy contribution in this direction was made by Heider and Simmel (24), who developed a silent filmed social episode with geometric figures in place of human actors. The technique seemed to offer a means of studying the perception of social interactions without contamination by the effects of gestural and facial expressions, clothing, etc. When the film was presented with varying instructions, it was shown that individuals differed considerably in the extent to which they attributed human values to the actions of the figures on the screen. Shor (39) has pursued this research.

In an independent development, film began to supplant apparatus in the experimental laboratory. Early experimentalists had evolved several varieties of stimulus presentation apparatus, including tachistoscopic devices using photographic materials. Film was already being used for laboratory demonstrations, and it was a relatively small next step, following Pillsbury's (32) suggestion, to the film recording of nonsense syllables and other kinds of meaningless materials for temporally controlled presentation to *Ss* in learning, memory, and perception experiments (Metfessel and Warren [31]; Ford [14, 15]; Beck [2]). A modern extension of these applications may be found in the Averbach and Coriell (1) study of short-term visual memory, using television to obtain sequential tachistoscopic effects.

With these developments in the 1930's, some psychologists began looking toward perceptual research for new approaches in personality measurement. An objective apparatus test of color *vs.* form dominance, using the principle of apparent movement, had been developed by Schmidt (35). Thurstone (42) modified Schmidt's rather complex tachistoscopic presentation, later referring to it as ". . . the most ingenious test that I have ever seen" (45, p. 5), and included the test in his factor analytic studies of perception (43), noting that still another version of the Schmidt test had subsequently been produced as a motion picture. The film used colored forms that appeared to move around a clocklike circle in opposite directions, with direction dependent on whether attention was directed at the forms or at their colors. Thurstone (45) also described further modifications of the test to take account of various kinds of direction dominance. While the test was limited to individual administration, it apparently represented for Thurstone a major step toward the objective measurement of temperamental characteristics.

Prior to the 1940's, developments in the use of film as tests are of historical interest but include no attempts to investigate the general uses of motion pictures as group tests. For the most part, the early studies represent a rather restricted view of film capabilities. Film was used only in its simplest and most conventional form, either as a record of human behavior or as a convenient "container" for static information which might have been communicated as well by means of apparatus.

The "modern" use of film in group testing originated during World War II; more specifically, it originated in the spring of 1942 and grew from the necessities of the War. The responsible individuals were J. J. Gibson, J. C. Flanagan, and F. A. Geldard.[2] Subsequently, Gibson, together with R. M. Gagné and other members of the Perceptual Research Unit (later the Psychological Test Film Unit), AAF Aviation Psychology Program, conceived, designed, and produced 21 motion picture tests. These must be regarded as "modern" since they typically utilized the unique capabilities of film in developing and presenting items unlike those that might have been presented in conventional printed form. These tests were concerned primarily with aptitudes which might be involved in motion and distance perception. Frequently, they represented complex, sequential, and dynamic identification and discrimination tasks. Several of the films made use of animation techniques and/or systematically varied item exposure timing, stimulus motion, or rate of movement. Many also used subjective camera angle and mock-ups to achieve simulation of tasks faced by aircrew members. Gibson's (17) summary presentation of these developments represents the most important, and the most general, discussion of motion picture testing possibilities to date. While the concurrent, and apparently independent, work of Thurstone, Ray, and Heider and Simmel has contributed useful ideas, it is the Air Force research that must be considered germinal. It provided much of the original impetus for current interest in cine-psychometry.

In addition to the studies reported by Gibson, several of the Air Force film tests have seen additional use. Roff (34) included nine of these instruments in a large factor analysis of Air Force printed and apparatus tests of perceptual skills. Two of the 18 factors reported were composed predominantly of film tests. Fruchter and Mahan (16) followed with a further analysis of these same tests, qualifying the nature of one of Roff's film test factors. Several tests have also seen use in Fleishman's (13) factor analytic work in psychomotor abilities.

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[2] Gibson, J. J. Personal communication, November 18, 1964.

There are other more recent developments concerning psychological measurement via cinematography which should be noted for this historical record, though they are of less relevance for the purposes of this paper than work previously cited. Schur (36) first suggested that printed test questions could be spliced into live action film to serve various educational measurement purposes. Curtis and Kropp (12) have contributed a substantial body of research on the characteristics of printed items presented via timed slides or filmstrips, and Heckman (23) has recently added further to this literature. A live action, itemized film test measuring proficiency of Army mechanics was developed and analyzed by Carpenter *et al.* (8) who provided a general discussion of film test applications in proficiency measurement.

Relative motion problems have been presented by film in research conducted by Bryan and Rigney (6). Johnson and Vogtmann (25) described their use of commercial film clips as part of a course examination in psychology. Cline (9) has used filmed stress interviews as stimulus material for the measurement of interpersonal judgment abilities. Similarly, interviews on film were used by Stoller and Geertsma (40) to test the clinical judgment of student psychiatrists. Motion pictures have also been used to present a kind of "dynamic" Rorschach (26, 27) and to simulate reality in the Thematic Apperception Test. McIntyre (28, 29) has summarized this latter literature. Lastly, Van Horn (48) has converted the Guilford-Zimmerman Temperament Survey to live motion picture form.

CURRENT DEVELOPMENTS

Research recently completed by the authors and further investigations now under way (37, 38) have sought to bring several of the trends identified above together in factor analytic studies of visual and auditory cognition and memory.[3] These studies were planned as factor analyses because they were frankly exploratory, because they held promise for the isolation of unique abilities, and because the controlled versatility of film provided for the correlational analysis of the effects of manipulable, and traditionally experimental, stimulus conditions in terms of individual differences. It was reasoned, first, that many of the Air Force tests were not "merely" perceptual tasks. They involved sequential cognitive and accumulative memory processes which deserved to be considered in the light of present conceptions of intellectual organization, such as Guilford's Structure of Intellect (21).

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[3] The studies cited are being supported by grants from the Department of Health, Education, and Welfare, Office of Education, under the provisions of Title VII of the National Defense Education Act of 1958.

Secondly, although fidelity of report research, particularly Ray's work, has shown clearly that group-administered printed tests following complex motion picture presentations are feasible and demonstrate marked individual differences, factor analytic studies of memory have not included such measures. Filmed human action, and possibly abstracted social situations like those developed by Heider and Simmel, might also be of use in exploring Guilford's behavioral content column. Thirdly, tasks typically used in experimental investigations of perception and memory can be executed on film for presentation to large groups. They deserve to be so executed both because experimental psychologists have usually ignored individual differences in such work and because the underlying nature of ability factors, long a controversy fueled without benefit of *experimental* analysis, might well be clarified thereby. These studies have ignored, for the present, possibilities for the objective measurement of temperament as well as several of Guilford's intellectual domains.

The purposes of the initial investigation were (1) to begin the exploration of individual differences in performance on tasks characterized by moving, sequential, and/or behavioral content; and (2) to examine empirically a conception of factors as information processing operations. The core reference for the study was defined as the visual, figural, cognition-memory portion of Guilford's model, but it was hoped that base-line data could be obtained which would suggest possibilities for similar studies throughout the Structure of Intellect matrix.

A total of 28 experimental film tests were constructed, some with several subparts. These tests frequently involved moving geometric figures or sequences of static figures, letters, photographed objects, or parts thereof. Some were based on recordings of real human actions or environments. Typically, the forms of response required of subjects did not differ from those usually involved in printed cognition and memory tests. Most tests called simply for recognition or recall of film content. However, reasoning involving a temporal sequence of figural changes or transformation of a temporally sequenced display into alternative forms was incorporated into some tests. Eight of Gibson's film tests, along with the newly produced film tests, were combined with printed reference measures representing known dimensions and were administered to 100 freshman engineering students at Purdue University. Analyses of these data have permitted the identification of several new factors that involve integration or closure of temporally spaced percepts into meaningful wholes, accumulation and isolation of significant details in such serial presentations, and various kinds of cognitive manipulation of remembered spatial and temporal relations. The work has provided additional factors which must be left unidentified for the present, as well as many hypotheses for further factor analytic studies.

More important than the isolation of new factors, however, are implications concerning the informational nature of intellectual performance. Guilford has previously emphasized this informational approach and has suggested that factors be interpreted as mental functions (22). In some respects, the data might be taken to suggest extensions or modifications of Guilford's structure model according to assumed underlying information-processing mechanisms. First, visual imagery seems better conceived as a strategy than as an ability dimension. An individual faced with a memory task, particularly one that is left unstructured for him by test instructions and/or that requires rapid storage of complex sequential information, may use iconic representation (i.e., imagery) for storage and a kind of "inner video tape recorder" for playback. For other memory tasks, even apparently nonverbal ones, he might apply symbolic representation (i.e., verbal abstraction) as information is received. The crucial stimulus variables would seem to be the nature of instructions, the degree of complexity of the information involved, and the speed with which the information must be handled.

Second, the serial nature of several obtained factors suggests that a discrete dichotomy between visual cognition abilities and visual memory abilities, while helpful in discourse, is in a sense a convenient fiction. There should be an underlying time continuum along which the various phenomena of sensation, perception, cognition, and memory may be arrayed. Because visual input must travel this path and because such input is a complex stream of information, a sequence of interdependent factors can be expected which represent individual differences in reception, integration, accumulation, and storage operations. This conception of an operation series is related to the two-stage memory system suggested by Averbach and Coriell (1) and by Broadbent (5). Tests presenting treatment conditions similar to those used by Averbach and Coriell were included here. Thus, a column of serial cognition abilities was hypothesized to exist between Guilford's cognition and memory columns and some of these factors were found. Third, just as the existence of operation series across cognition and memory seems plausible, there is some justification for viewing Guilford's product dimension as an operation hierarchy. A second-order factor was obtained which seemed to suggest, for the cognition of figural content at least, that tests at the units level involve simple configuration, while tests representing higher level products require more complex internal manipulation of lower level configurations.

Such results are tentative and incomplete at present, and further analyses of the initial data plus a relatively extensive second study are planned. In addition to its confirmatory role, the second stage of this research is concerned more exclusively than was the first with meaningful memory, that is, with Guilford's semantic content category. Its purposes are (1) to explore the factors of human auditory and visual memory for meaningful information; and (2) to determine the extent to which such factors account for individual differences in auditorially, visually, and audiovisually communicated formal instruction. It is hoped that the research will lead to continued development of a conception of abilities as information-processing mechanisms, contributing to the understanding of audiovisual communication along lines similar to those recently discussed by Travers (46). It seems clear at this point, however, that the degree of stimulus control provided by film, when incorporated into traditional factor analytic research on intellect, can contribute uniquely to the study of human cognitive processes.

SOME CONCLUDING SPECULATIONS

In spite of the fact that films and their use in psychological testing cover a significant span of years, it is clear that the volume of work has been small and that the future is correspondingly difficult to predict. Nevertheless, it is not unreasonable to suppose that there will be a future. This supposition is not merely a matter of author preference, nor is it dependent only on the unsatisfied suggestions of Thurstone, Guilford, and Gibson. Instead, the available evidence, limited as it is, also argues effectively in favor of continued work. The purposes to be served are in part practical, yet the larger opportunities seem to be those associated with expansion of present work into the nature or dimensions of human intellect. If some readers are inclined also to see important contributions which might be made to the further study of human temperament or "personality," the authors would not disagree but would comment only that those areas of study lack the clarity that has appeared from studies of intellect and are somewhat more difficult to project forward in time.

Among the practical applications, future work should surely recognize the testing needs that are peculiar to special or atypical populations. In particular, it would seem that opportunities will be great wherever examinee populations reflect marked deficiencies in one or several language skills, e.g., the deaf or hard-of-hearing or the culturally deprived. Other populations for which useful applications might be made are aphasics, the retarded, and those with primary backgrounds in a noncommon language.

It is also worth noting that technology has now provided several serviceable 8mm projectors, both sound and silent, and these, or later versions of them, seem promising for use in individual or small-group testing. Since these projectors also provide for easy cartridge loading and are generally uncomplicated in operation, they are particularly reasonable to consider. Perhaps brief subtests could be prepared, each in a single cartridge, and could then be administered almost automatically. This kind of freedom from the more cumbersome group administrations of tests prepared in 16mm film is worth pursuing.

Apart from the practical applications, there is the continuing prospect of expanded work in the study of human intellect. Gibson, as already quoted, strongly suggests the contributions that might be made; Guilford has been inclined to agree at least in part; and Thurstone indirectly notes similar possibilities. In addition, results from the authors' study provide further support, even though these results must be described now as tentative. Within the study, at least two ability factors (currently labeled "temporal closure" or "perceptual integration" and "configural accumulation") are heavily dependent on film tests.

Furthermore, the study appears to provide evidence that conforms to major ideas of Piaget and/or Bruner. These appearances stem from two highly similar "live action" film tests that were administered adjacent in time and were preceded in each case by vague instructions; examinee responding consisted of answering several questions concerning the film contents and the portrayed events immediately after each film had been shown. The first of these tests appears in a factor composed of perceptual and visual scanning (nonverbal) tests, while the second appears in a factor with many customary measures of verbal abilities. From this, it can be suspected that examinees tended at first to process iconic or "eidetic" images, but shifted noticeably toward a verbal labeling strategy on the second test, once the nature of their task seemed predictable. If results such as these are repeatable, they would suggest again that iconic or eidetic schemes are regularly used by the young until language skills become developed. Then, in later years, they tend to abandon these schemes and to resort instead to selective verbal labeling of visual experience. So, even though the results above are only a sample of tentative findings, they should serve to justify continued and expanded investigations.

In an earlier quotation from Thurstone, he refers to Schmidt's color-form dominance test as "ingenious"; few would disagree. It can be suggested, however, that film offers many opportunities for the ingenious test to become commonplace. Even in the absence of color and third-dimensional cinematography, the variable combinations available for use in item development are just less than limitless. Item content may, in Guilford's terms, be figural, symbolic, semantic, or behavioral. Both the auditory and the visual channels may be used, either competitively or as complements, either alternating or concurrently, with similar contents or with different.

Exposure intervals may be tachistoscopic (the exposure duration of a single film frame is approximately equal to 21 milliseconds), intermediate, or extended. The time interval between stimulus exposure and response selection can be made as brief or as long as desired. Similarly, the time allowed for response selection or decision making can be ample or abbreviated; investigators who are interested in the experimental inducement of anxiety might find that repeated reductions in response time could help in the indexing of performance under increasing pressure. In addition to controlling the intra- or inter-item time intervals and, thus, controlling the pace of stimulus exposure, motion and rhythm are two manipulable time-based variables which further expand the opportunities that are open to the test constructor. Few simple effects attributable to the use or manipulation of these variables are expected, but many effects in which they are involved seem likely.

It is hoped that future work can expand in several other directions also. There is already considerable literature relating to the abilities involved in the processing of print and other predominantly verbal material, but little is known concerning the particular abilities which may equip a learner to deal effectively with the dynamic and sometimes nonverbal materials of film, television, and other pictorial media. These abilities, which might be called "cinemacy" (as opposed to "literacy"), merit study. In the process of studying them, it is possible not only that something more can be learned about the special and functioning abilities of learners, but also that the features of pictorial media and the ways in which they influence learners can be better understood.

Earlier, brief mention was made of the detailed and synchronized responses that might be gathered in addition to the customary gross responses that serve only to indicate whether an examinee has answered an item correctly or incorrectly. At the risk of offending some film or psychometric experts, we would propose the occasional usefulness of examinee response latency measures (41), of data which index the visual scanning patterns of individual examinees (18), and even of electroencephalogram records (49). Examinees do in fact require time to process item stimulus materials and to arrive at or select an answer. We should therefore be well advised to learn more than can presently be said about the relationship between processing time, response adequacy, item difficulty, and the many other variables of interest either to the investigator of individual differences, of learning, or of both.

Similarly, the visual scanning strategies employed in identifying critical elements within a visually presented problem seem valuable to determine, not only for assessing the simpler matters of temporal order and cues utilized, but also for relating tests within a factor to those within the same or different factors. Finally, the Vogel and Broverman article (49) suggests that there may indeed be some substantial relationships between measured abilities and the electrocortical measures provided by the EEG. If such relationships are to be established and refined, the combined use of controllable film test items and EEG opens some opportunities for studies which previously were difficult to conceive.

Finally, some who know of Guilford's work are inclined not to accept his organization of multiple abilities and, particularly, his predictions of still further abilities that may be expected to emerge. For them, the predicted number of differentiable ability factors is too great. The authors are reluctantly inclined to suspect the reverse, i.e., the emergence of even more differentiable factors than Guilford has proposed. An elaborate argument in defense of this suspicion cannot be attempted, yet there are grounds for it both in the factors that seem already to be emerging from the dynamic and sequential film items and in the greater variety of novel tests, not yet prepared, that are even farther removed from conventional tests than those employed to date. The proliferation of ability factors is neither desirable nor undesirable. It may be justifiable, however, and only time will tell.

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